Name: _____

Class Time (circle one)   10   11   12

Final Examination

## Mathematics Xa

January 15, 1998

Problem	Points	Score
1	7	
2	10	
3	9	
4	7	
5	6	
6	9	
7	12	
8	8	
9	11	
10	11	
11	10	
Total	100	

Please show all your work on this exam paper. You must show your work and clearly indicate your line of reasoning in order to get credit. If you have work on the back of a page, indicate that on the exam cover.

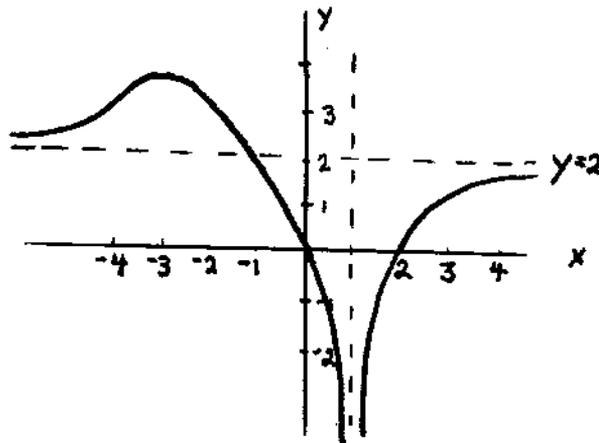Give exact answers except when an approximation is requested.
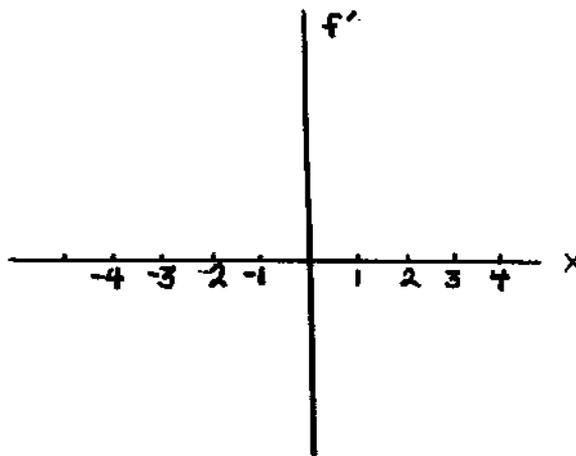You have three hours for this exam. Work carefully and efficiently. Think clearly, and do well!
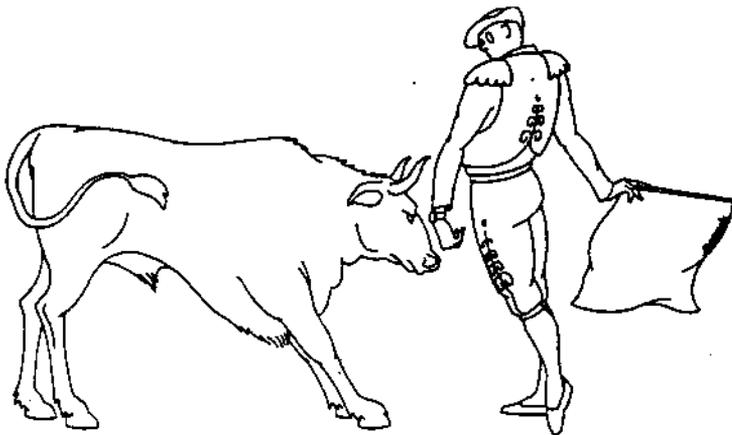
Use your time wisely!   Bartak/Nebelspalter/Zurich

1

1. (7 points) The graph of $f(x)$ is given below.

(a) On what x-interval(s) is $f'(x)$ negative?

(b) On what x-interval(s) is $f''(x)$ negative?

(c) Where on the interval $(-\infty, 1)$ is f' biggest?

(d) On the axes provided, graph $f'(x)$.

2. (10 points) Differentiate the following.

(a) $f(x) = \dfrac{\pi}{\sqrt{2x^\pi + x}}$

(b) $f(x) = \dfrac{1}{\ln(x^2 + 1)}$

(c) $f(t) = \dfrac{5 \cdot 2^t \cdot t^7}{4}$

(d) $f(x) = 3\ln\left(\dfrac{x\sqrt{x}}{5x + 1}\right)$

3. (9 points) Let $g(t) = \dfrac{t}{t-1}$

    (a) Find the average rate of change of g on the interval $[1, 4]$.

    (b) Use the limit definition of derivative to find the instantaneous rate of change of g at $t = 4$. You do not need to find $g'(t)$, just $g'(4)$. Show all your algebraic manipulations clearly. We are interested in your write-up, not just your answer, so pay particular attention to how you express yourself.

    (c) How much confidence do you have in your answer to (b)? Explain your reasoning.

4. (7 points) Rank the following investments from best to worst according to their gain over a **two** year period. List the best one first, the second-best next, until you have ranked all from best to worst. Show your work.

   (a) Investment A grows 12% each year.

   (b) Investment B grows 25% in two years.

   (c) Investment C grows 1% each month.

   (d) Investment D doubles every 6 years.

   (e) Investment E grows 90% in the first year and then declines 34% in the second year.

Ranking: _____

5. (6 points) Sketch a graph of a function f with domain $(-\infty, \infty)$ having *all* of the following characteristics.

- f is continuous for all x.
- f has no roots.
- $\lim\limits_{x \to \infty} f(x) = 7$.
- The graph of f' is shown below.
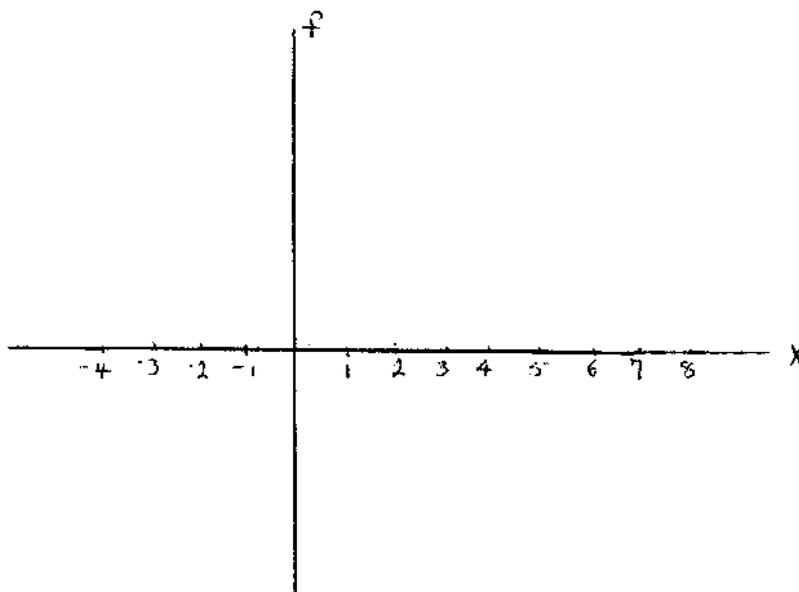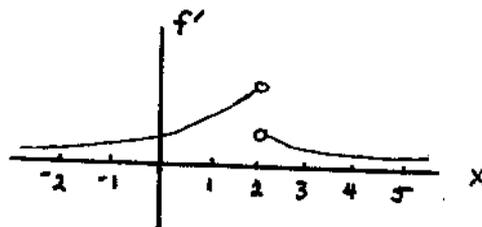
6. (9 points) Suppose that $p(x) = [h(g(x)) + \pi]^3$

where the functions g and h are both differentiable on $(-\infty, \infty)$.

(a) Find an expression for $p'(x)$ in terms of g, h, and their derivatives.

(b) Below is a table giving some information about g, h, g', and h' at various x-values. Find $\dfrac{dp}{dx}\Big|_{x=2}$.
Show your work.

	g	g'	h	h'
2	6	-3	21	-7
6	2	-4	13	-11
8	-4	-6	-9	-4

(c) Suppose you know that $h(x)$ and $g(x)$ are decreasing on $(-\infty, \infty)$. Which one of the following is true? Explain your answer thoroughly.

i. $p(x)$ is increasing on $(-\infty, \infty)$.

ii. $p(x)$ is decreasing on $(-\infty, \infty)$.

iii. $p(x)$ is increasing on some intervals and decreasing on other intervals.

iv. More information is necessary in order to decide among the choices above.

7. (12 points) A clinic has $22,000 to spend each month on paying doctors and nurses. Each doctor receives $6,000 per month and each nurse receives $2,000 per month. Suppose that the number of patients treated each month is proportional to the product of the number of doctors and the number of nurses.

How should the $22,000 be divided so that the greatest number of patients can be treated? (Fractional amounts of doctors and/or nurses are possible because people can be hired on a part-time basis.)

Although you are free to use your calculator to check your work, we are interested in your reasoning and require that your arguments be clear, complete, and stand completely independent of your calculator. Your answers must be exact. Be sure to justify that the answer you give actually maximizes the number of patients that can be treated each month.

8. (8 points) Find possible equations for each of the graphs given below.

a)
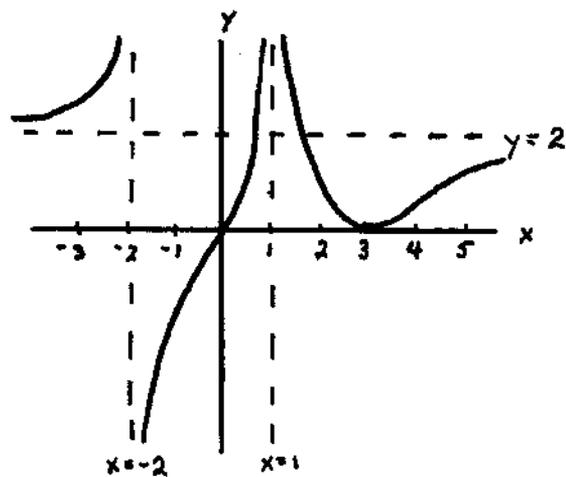
b)
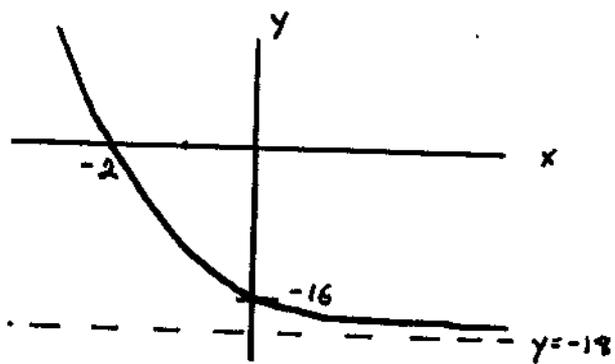
9. (11 points) Let $f(x) = x^3 e^{-bx}$ where b is a positive constant.

(a) Where does the graph of f cross the x-axis?

(b) Find all critical points and classify each as a local maximum, a local minimum, or neither.

(c) Does this function have a global maximum on $(-\infty, \infty)$? If so, what is its value? If not, explain why not.

(d) Does this function have a global mimimum on $(-\infty, \infty)$? If so, what is its value? If not, explain why not.

10. (11 points)

Part 1. The pollution level in a certain river has been increasing dramatically. Denote the pollution level at time t by $P(t)$ where $t = 0$ is the year 1920 and t is measured in years. Then the pollution level in the river can be modelled by

$$P(t) = P_0 e^{kt}$$

where P_0 and k are positive constants. We want to convey this information graphically, but due to the enormous range of $P(t)$ values, we will plot $\ln[P(t)]$ on the vertical axis, NOT $P(t)$.

(a) If $P(t) = P_0 e^{kt}$, then $\ln[P(t)] =$

The following questions are about the graph of y versus t, where $y = \ln[P(t)]$. They are not asking about the graph of P versus t.

(b) If $y = \ln[P(t)]$, then the graph of y versus t will be
   i. a straight line
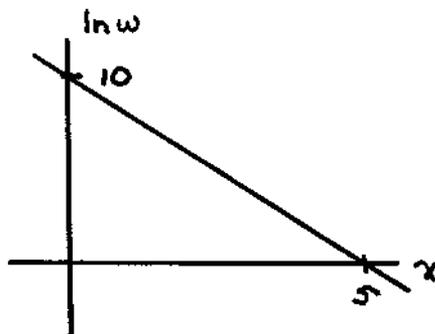   ii. a parabola
   iii. a cubic
   iv. none of the above

(c) What is the y-intercept of the graph?

(d) Under what circumstances will this intercept be negative? zero? positive?

(e) What is the slope of the graph of y versus t? Is it constant? .

Part II

(a) Suppose the graph of $\ln w$ versus x is the straight line shown below. Write w as a function of x.

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11. (10 points) Consider a parabolic bowl of milk. (The bowl is drawn below.) $h(V)$ is the function that takes as input the volume of milk in the bowl and gives as output the height of the milk. The domain of h is $(0, V_T)$ where V_T is the volume of the bowl. For each of items (a)-(d) below, select one of the answers and explain your reasoning carefully in words. Your explanation should refer to the volume and height of the milk. Let V_1 be half the total volume of the bowl.
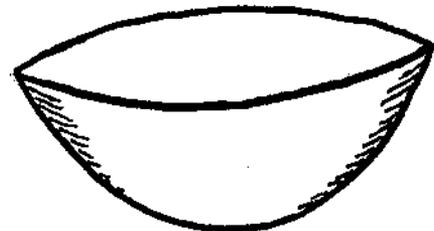
(a) $h(V_1)$ is

    i. positive

    ii. negative

(b) $h'(V_1)$ is

    i. positive

    ii. negative

    iii. zero

(c) $h'(V)$ is

    i. increasing

    ii. decreasing

    iii. constant

(d) $h''(V_1)$

    i. positive

    ii. negative

    iii. zero

(e) What is the meaning, in plain English, of $h^{-1}(1)$?