Getting Ready for the Final Part One!

Math X
Jan '01

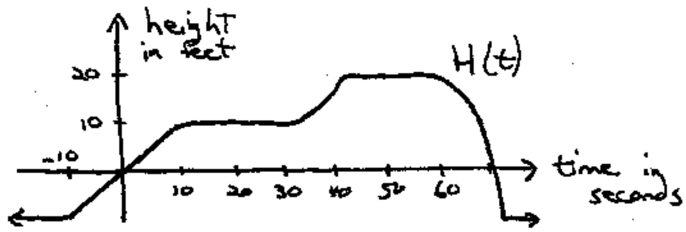
We'll split up the material over 3 review sessions:

Topics:

A. Words to graphs; words to math
B. Definition of derivative
C. Going from $f \to f'$; $f' \to f$; etc.
D. Limits - analytically and graphically
E. Functions: Polynomials
Exponentials
Logarithms
F. Extrema - critical points; local/abs max/min
G. Flips, shifts, stretches of functions
H. Inverse functions (graphs and word explanations)
I. Differentiation Part 2 - differentiating basic functions
- Product / Chain Rule
J. Solving Equations using logs, exponentials
K. Optimization problems

(A) Words to graphs; etc.

1)



Suppose $t = 0$ represents 5 pm today and $H(t)$ gives the height of an object above ground level (positive = above ground)

a) When is the object's velocity equal to 0?

b) When is the object's speed maximum?

c) How far away from the ground does the object get?

d) What might the object be?

2) Suppose $I(t)$ gives the amount of ink left in my pen at time t, where $t=0$ represents noon, and t is measured in minutes, and $I(t)$ is measured in milliliters

What do each of the following mean?

a) $I(-30) = 5$

b) $I(0) = 5$

c) $I(20) = 3$

d) $|I'|$ is maximal at $t = 15$

e) $I'(35) = 0$

f) $I(70)$ is an absolute minimum for $I(t)$

g) $I(100) = 0$

How would you write an equation to show

h) I have no ink left after one o'clock

i) The pen holds 10 milliliters of ink, and it was full before 11 am

B) Definition of the Derivative

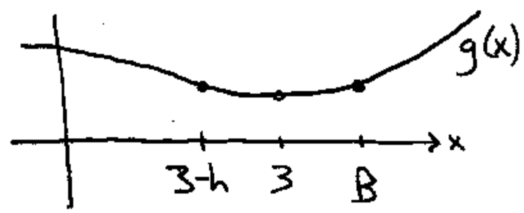
3) Using the usual definition $f'(x) = \lim_{h \to 0} \Big(\qquad\qquad \Big)$

find $f'(x)$ if $f(x) = \frac{10}{x}$

b) find $f'(3)$

c) find $f'(0)$

4) Find $g'(x)$ if $g(x) = 3x^2 - 2$ (using the limit definition again!)

5) 

Which of the following equals $g'(3)$?

a) $\lim\limits_{B \to 0} \dfrac{g(B) - g(3)}{B}$

b) $\lim\limits_{h \to 3} \dfrac{g(3) - g(h)}{h}$

c) $\lim\limits_{B \to 3} \dfrac{g(3+B) - g(3)}{3+B}$

d) $\lim\limits_{h \to 0} \dfrac{g(3-h) - g(3)}{-h}$

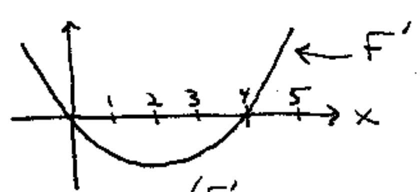
e) $\lim\limits_{h \to 0} \dfrac{g(3-h) - g(3)}{h}$

f) $\lim\limits_{B \to 3} \dfrac{g(B) - g(3)}{B-3}$

g) $\lim\limits_{B \to 3} \dfrac{g(B-3) - g(B)}{3}$

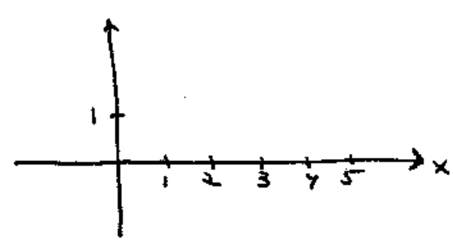
h) 0

C) Going from $f \to f'$, etc.

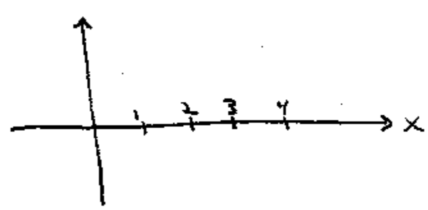
6) Given 

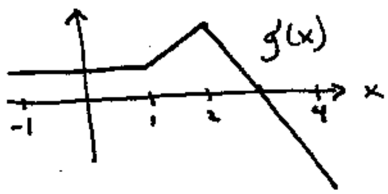
(f' is a parabola)

draw a possible graph of f if $f(0) = 1$

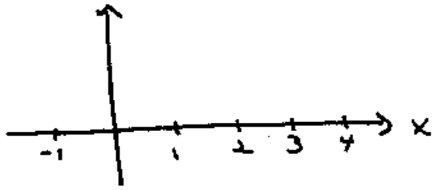


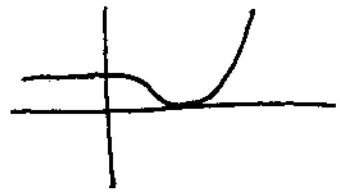
graph f''



7) Given 

graph a possible $g(x)$ if $g(0)=0$ 

graph $g''(x)$ 

8) Graph f and f'' if you're given f': 
again let $f(0)=0$

(D) Limits

9) What is $\lim_{x\to 0} (x^3+5x-10)$?

10) If $f(x) = 3x^3-2x$ (a) what is $\lim_{x\to\infty} f(x)$?

(b) what is $\lim_{x\to 0} f'(x)$?

(c) what is $\lim_{x\to -\infty} f''(x)$?

11) If $H(x) = \left(\frac{4}{x^2}+\frac{3}{x-3}+1999\right)$

(a) then what is $\lim_{x\to 0} H(x)$?

(b) what is $\lim_{x\to 2} H(x)$?

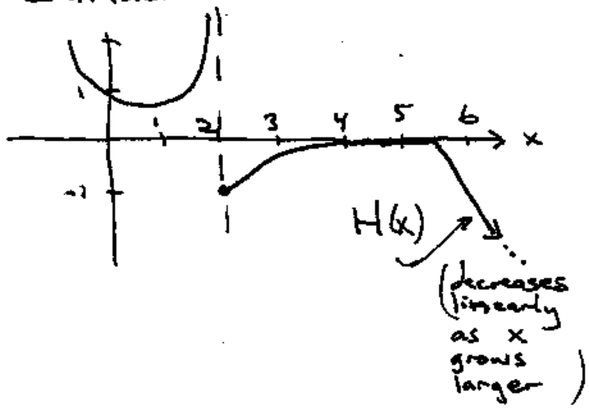
(c) $\lim_{x\to\infty} H(x)$?

(d) $\lim_{x\to -\infty} H(x)$?

12) What is a) $\lim_{x\to -\infty} 1999x^{2000}$? b) $\lim_{x\to -\infty} 1998x^{2001}$?

Limits continued

13)



(a) What is $\lim_{x\to 2^+} H(x)$?

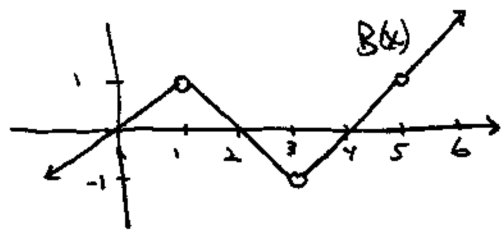
(b) What is $\lim_{x\to 2} H(x)$?

(c) What is $\lim_{x\to \infty} H(x)$?

(d) What is $\lim_{x\to 5^+} H(x)$?

(e) What is $\lim_{x\to 5} H'(x)$?

14)



a) What is $\lim_{x\to 1} B(x)$?

(b) What is $\lim_{x\to 2} B'(x)$?

(c) What is $\lim_{x\to \infty} B'(x)$?

(d) What is $\lim_{x\to 5} B(x)$?

(e) What is $\lim_{x\to 1} B'(x)$?

(f) What is $\lim_{x\to 5} B'(x)$?

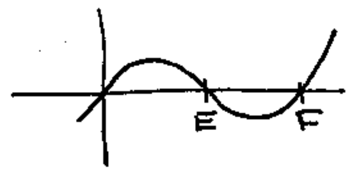
E) Functions - Polynomials

Remember if you can factor it then you should probably go ahead (if it's easy)

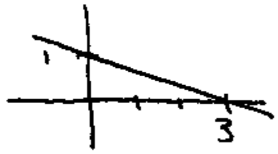
15) What are the roots of $(x-3)^2(x^2-25)$

16) Sketch a graph of $x(x-2)(x+2)$

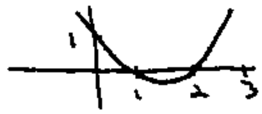
17) What is a possible function for



18) Give the function for



19) Give a possible function for



20) If $B(x) = 3x^5 + 2x^3 - p \cdot x + 27$

a) What is $\lim_{x \to \infty} B(x)$?

b) $\lim_{x \to -\infty} B(x)$

c) What is the slope of $B(x)$ at $x = 0$?

d) What is $B(x)$'s y-intercept

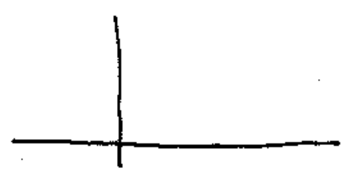
e) What is the maximum # of roots of $B(x)$?

f) What is the maximum # of turning pts of $B(x)$?

g) Does $B(x)$ have an absolute max. or min.?

Exponential Functions

Know basic shape 21) graph $3^x + 2$



22) If the number of red pens, $R(t)$, is 24 at time $t=0$ (t measured in weeks) and pens decrease at a rate of 25% every two weeks, give a formula for $R(t)$.

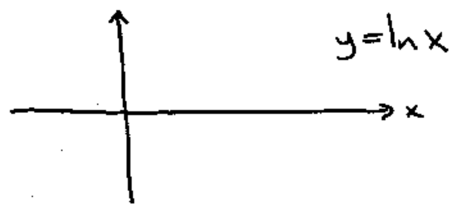
Final Push for the Final Part Two Math Xa

continuing the series of topics:

E. Functions – Logarithms
F. Extrema – critical pts, abs/loc max min
G. Flips, shifts, stretches of functions
H. Inverse functions

E. Functions – Logarithms

know shape of logarithm graph:



and basic properties:

1) $a^b = x$ is the same as

$\log_a \_ = \_$

2) $\ln(a \cdot b) =$

3) $\ln\left(\frac{a}{b}\right) =$

4) $\ln\left[w^z\right] =$

5) Simplify the following:

a) $2\log x - \log(3x) + 2\log(\sqrt{3})$

b) $\log_3 27 + \frac{1}{2}\log_{12} 144 - \log(1{,}000{,}000)$

c) $\ln(e^x) - 3\log_5(25) + \log_{100} 10$

d) $\log 15 - \log 3 - \log(\ln(e^5))$

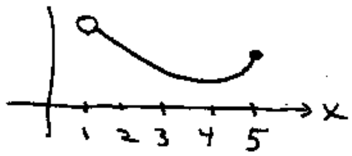
(F) Extrema!

Know Definition of Critical point:

Critical Points are the values which <u>might be</u> Extrema

Defn of Extrema:

6) If $g(x)$ looks like (only defined for $1 < x \leq 5$)



What are $g(x)$'s Extrema?

7) How about $g(x) = x^3 - x$? What are its Extrema, and what are its critical points?

8) Use the second derivative test to check if $h(x) = \frac{1}{3}x^3 + x^2 + x$ has a local max or min. (first find any critical points!)

9) What does the first derivative test say about the extrema for $B(x) = x^2 - 6x + 9$?

G) Flips, Stretches, etc.

Given $y = f(x)$ what happens to the graph if...

10) $y = 3f(x)$

11) $y = \frac{1}{2}f(x)$

12) $y = -f(x)$

13) $y = f(x) + 5$

14) $y = f(x-2)$

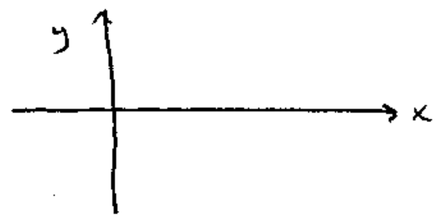
15) $y = f(x+3)$

16) $y = f(-x)$

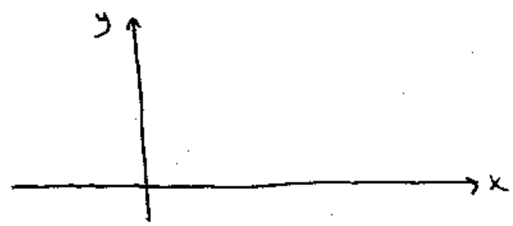
17) $y = f(5x)$

18) Graph a) $y = (x-3)^2 + 2$

b) $y = -\ln(x^2)$



c) $y = 3e^{-x} - 1$



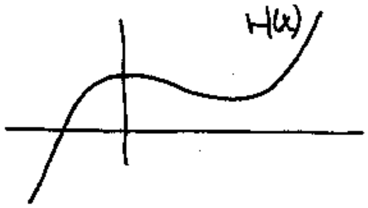
d) $y = \frac{1}{2}(2^x) + \frac{1}{2}$

(H) Inverse Functions

Know the trick for finding inverses algebraically

19) If $F(x) = \frac{x+2}{x-1}$ then what is $F^{-1}(x)$?

20) Where is $f(x)$ defined (what is its domain)?

21) Is  $H(x)$ invertible?

22) I can make $B(t)$ brownies in t minutes. On the day before the last class I made D brownies in C minutes.

Describe in words each of the following:

a) $B(30) = 65$

b) $B^{-1}(65) = 30$

c) $B^{-1}(230 + D)$

d) $B^{-1}(D - 25)$

e) $B(C + 10)$

f) $B(C) + 10$

g) $3(B^{-1}(D))$

Getting Ready for the Final - Part 3!!

Math Xa

Remaining topics include:

I. Differentiation - basic functions and Product/Quotient/Chain Rules

J. Solving Equations - using logs/exp.s

K. Optimization Problems

I. Differentiation

Know the derivatives of all the basic functions (polynomials, powers of x like $\frac{1}{x^2}$, exp.s, logs)

Differentiate each of the following functions

1) $7x^4 - 2x + \frac{1}{x}$

2) $ax^3 + be^x$

3) $12e^x + \log_{12} x$

4) $5 \cdot \ln(6x^5)$

5) $\frac{1}{w \cdot x^2} - \frac{1}{z \cdot (x^{-1})}$

6) $2^{(4x)} - 4^x$

Recall the Product Rule:
the derivative of $f(x) \cdot g(x)$ is ?...

Find the derivative for each of the following
(I didn't know it was lost!)

7) $(2x+3) \cdot (e^x)$

8) $\left(\ln (x^5)\right) (5^x + 3x)$

9) $(12x^5 + 5)(x^3 - 1)$

Quotient Rule: recall that the derivative of $\frac{f(x)}{g(x)}$ is ?...

Find the derivative of

10) $\frac{3x^2}{5x+2}$

11) $\frac{2^x}{3^x + 5^x}$

12) $\frac{\ln (x^4)}{\log_{12} (x^4)}$

Finally, Chain Rule recall that the derivative of $f(g(x))$ (i.e. a composition of functions $f(x)$ and $g(x)$)

is ?...

So find the derivatives for each of the following composition functions

13) $(3x+12)^5$

14) $(\ln(x)+1)^{98}$

15) e^{kx}

16) $\dfrac{1}{e^{\log_5 x}}$

17) $H(14x+2)$

18) $H(5x^3+e^x)$

J) Solving Equations using Logs and Exponential functions

- pull down powers involving x by taking logs
- undo logarithms by taking exponents

Solve each of the following for x (solve exactly, not just numeric approximations)

19) $5^{2x+3} = 27$

20) $\ln x = 3$

21) $8^x = 3 \cdot 12^x$

22) $\log_{15} x - \log_{15} w = \log_{15} z$

23) $e^{3x} \cdot e^{-2x} = e^{\log_5 19}$

24) $\log_{12} x^2 + \log_{12} x^3 = 5$

(k) Optimization Problems

1) Name variables you need (width $= w$, radius $= r$, etc.)
2) Find equations: a) linking variables if you've got more than one
   b) giving the function you want to optimize
3) Usual trick $\Rightarrow$ solve for one of the variables and substitute the result into the function you want to optimize
4) Optimize! (usually just find derivative, set $=0$, check max/min)

25) A family goes on vacation, and they decide to drive. Assuming that they need to drive 500 miles and their car uses up $.002 s^2 - .005 s + 2.5$ gallons per hour, where s is the speed of the car, then if they drive at a constant speed, then what should that speed be to minimize the amount of gas that they use?

Answers to Math X Review Sheets

Part One

1) a) velocity $=0$ when graph is horizontal, so $t<-10$ and $10<t<30$ and when $40<t<60$

b) speed $=$ absolute value of velocity (i.e. speed is always positive or 0). Speed is maximum right after $t=70$

c) 20 feet

d) an elevator, perhaps, stopping every 10 feet to let people on, and then dropping a little too quickly back to the basement at the end!

2) a) 5 millilitres of ink left in pen at 30 minutes before noon (11:30 am)

b) There are 5 milliliters of ink in the pen at noon $(t=0)$

c) At 12:20 there are 3 millilitres of ink in the pen

d) The rate of flow of the ink out of the pen is fastest at 12:15 pm

e) No ink is being used right at 12:35

f) No more ink is used after 1:10 $(t=70)$

g) All the ink is used up by 1:40 $(t=100)$

h) $I(x)=0$ for $x>60$

i) $I(x)=10$ for $x<-60$

3) a) $f(x)=\frac{10}{x}$, so $f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)=\lim_{h\to 0}\left(\frac{\frac{10}{x+h}-\frac{10}{x}}{h}\right)$

$$=\lim_{h\to 0}\left(\frac{\frac{x\,10}{x(x+h)}-\frac{(x+h)\,10}{(x+h)x}}{h}\right)=\lim_{h\to 0}\left(\frac{\frac{10x-10x-10h}{x(x+h)}}{h}\right)=\lim_{h\to 0}\left(\frac{\frac{-10h}{x(x+h)}}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{-10}{x(x+h)}\right)=\frac{-10}{x^2}$$

b) $f'(3)=\frac{-10}{3^2}=-\frac{10}{9}$

c) $f'(0)$ wow, it's undefined (trick question!)

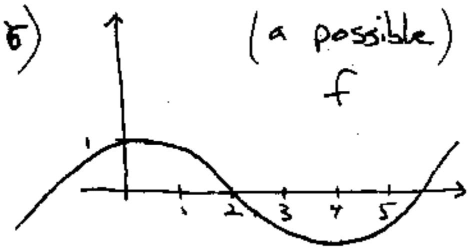
4) $g(x)=3x^2-2$ so $g'(x)=\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)=\lim_{h\to 0}\left(\frac{3(x+h)^2-2-(3x^2-2)}{h}\right)$

$$=\lim_{h\to 0}\left(\frac{3(x^2+2hx+h^2)-2-3x^2+2}{h}\right)=\lim_{h\to 0}\left(\frac{3x^2+6hx+3h^2-3x^2}{h}\right)$$

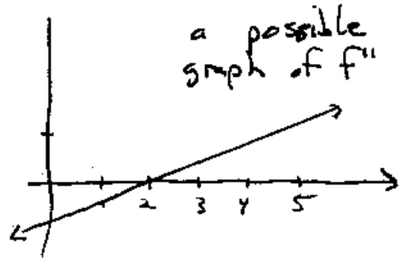
$$=\lim_{h\to 0}\left(\frac{6hx+3h^2}{h}\right)=\lim_{h\to 0}\left(6x+3h\right)=6x$$

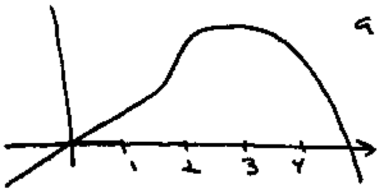
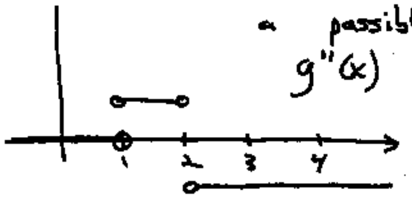
5) Just d, f and h

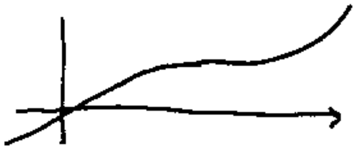
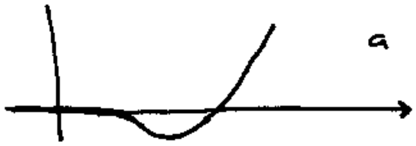
6) (a possible) f



a possible graph of f''



7)  a possible $g(x)$  a possible $g''(x)$

8)  a possible $f'(x)$  a possible $f''(x)$

9) -10

10) a) $+\infty$ b) $f'(x) = 9x^2 - 2$, $\lim_{x\to\infty} f'(x) = -2$ c) $f''(x) = 18x$, $\lim_{x\to-\infty} f''(x) = -\infty$

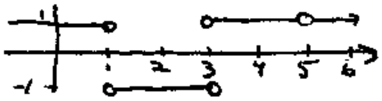
11) a) $+\infty$ (graph the function to check) c) 1999

b) $\frac{4}{2^2} + \frac{3}{2-3} + 1999 = 1 - 3 + 1999 = 1997$ d) 1999

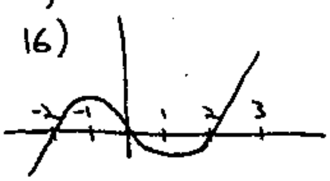
12) a) even exponent, positive coefficient $\Rightarrow \lim_{x\to-\infty} = +\infty$ b) odd degree, positive $\Rightarrow \lim_{x\to-\infty} = -\infty$

13) a) -1 b) undefined c) $-\infty$ d) 0 e) 0

14) a) 1 b) -1 c) 1 d) 1 e) undefined f) 1

$B'(x)$: 

15) roots at $x = 3$, $x = \pm 5$

16) 

17) $x(x-E)(x-F)$

18) slope $= -\frac{1}{3}$, y-intercept $= 1$, so $y = -\frac{1}{3}x + 1$

19) $y = k(x-1)(x-2)$ and $y = 1$ when $x = 0$, so $1 = k(-1)(-2)$ means $k = \frac{1}{2}$, so $y = \frac{1}{2}(x-1)(x-2)$

20) a) $+\infty$

b) $-\infty$

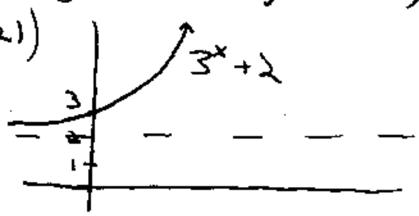
c) $B'(x) = 15x^4 + 6x^2 - p$, so $B'(0) = \text{slope at } x = 0 = -p$

d) 27

e) degree 5 $\Rightarrow$ max. of 5 roots

f) maximum of 4 turning points

g) odd degree (5), no absolute max or min

21) 

22) $R(t) = 24\,(.75)^{t/2}$

Answers for Part 2

1) $\log_a x = b$ 2) $\ln a + \ln b$ 3) $\ln a - \ln b$ 4) $z \cdot \ln(w)$

5) a) $2\log x - (\log 3 + \log x) + \log(\sqrt{3})^2 = 2\log x - \log 3 - \log x + \log 3 = \log x$

b) $3 + \frac{1}{2}(2) - \log_{10}(10^6) = 3 + 1 - 6 = -2$

c) $x - 3(2) + \frac{1}{2} = x - 5\frac{1}{2}$

d) $\log 5 + \log 3 - \log 3 - \log 5 = 0$

6) local, absolute min. at $x = 4$, no local or abs. max's

7) $g'(x) = 3x^2 - 1$, so $g' = 0$ when $x = \pm\frac{1}{\sqrt{3}}$ are the critical points.
$g''(x) = 6x$, and $g''\left(-\frac{1}{\sqrt{3}}\right) < 0 \Rightarrow x = -\frac{1}{\sqrt{3}}$ is a maximum

$g''\left(+\frac{1}{\sqrt{3}}\right) > 0 \Rightarrow x = +\frac{1}{\sqrt{3}}$ is a minimum

(2nd derivative test, concave up $\cup \Rightarrow$ minimum, etc)

8) $h'(x) = x^2 + 2x + 1 = (x+1)^2$ so only critical point is $x = -1$

$h''(x) = 2x + 2$, and $h''(-1) = 0 \Rightarrow$ 2nd derivative fails to tell us anything!

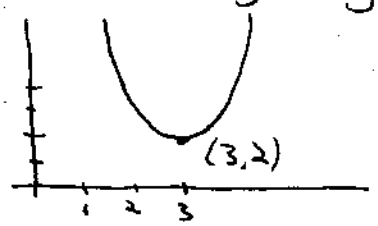
to check what's happening look at first derivative again: $h'(x) = (x+1)^2$ is always positive (or 0 at $x = -1$) so there aren't any local mins, maxs.

$x = -1$ is in fact an inflection point, since $h''(x) = 2x + 2$ changes from negative to pos. at $x = -1$.

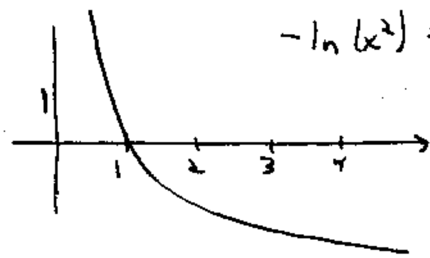
9) $B'(x) = 2x - 6$, so critical point at $x = 3$,
here $B'(x)$ goes from negative to positive,
so must be a local/absolute min. at $x = 3$

10) vertical stretch by 3 times
11) vertical shrink (down by $\frac{1}{2}$)
12) flip over x-axis
13) shift up by 5
14) shift to right by 2
15) shift to left by 3
16) flip over y-axis
17) horizontal shrink

18) a)

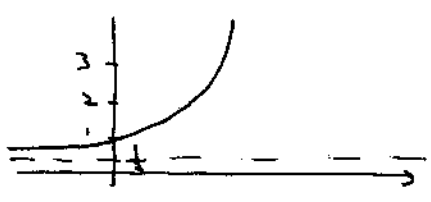


b) $-\ln(x^2) = -2\ln x$



c)

d)



19) first $y = \frac{x+2}{x-1}$, Then switch x, y $x = \frac{y+2}{y-1}$, solve for y

$x(y-1) = y+2$, so $xy - x = y+2$, $xy - y = x+2$

$y(x-1) = x+2 \Rightarrow y = \frac{x+2}{x-1}$

20) only not defined at $x = 1$

21) Not invertible, fails horizontal line test

22) a) I bake 65 brownies in 30 minutes

B: minutes $\to$ brownies

B^{-1}: brownies $\to$ minutes

b) It takes 30 minutes to bake 65 brownies

c) The time it takes to bake 230 more brownies than I made the day before the last class

d) The time it takes to bake 25 fewer brownies than I made the day before the last class

e) The number of brownies I could bake if I spent 10 more minutes baking brownies than I did the day before the last class.

f) $= D + 10$, 10 more brownies than I made the day before the last class

g) $= 3C$, Three times the amount of time I spent baking brownies the day before the last class.

Part Three

1) $28x^3 - 2 - \frac{1}{x^2}$

2) $3ax^2 + be^x$

3) $12e^x + \frac{1}{(\ln 12)x}$

4) $= 5(\ln 6 + \ln x^5) = 5\ln 6 + 25\ln x$, derivative $= \frac{25}{x}$

5) $\frac{-2}{w \cdot x^3} - \frac{1}{z}$

6) $\ln(16) \cdot 16^x - \ln(4) \cdot 4^x$

7) $(2x+3)e^x + 2e^x = (2x+5)e^x$

8) $\ln(x^5)((\ln 5)5^x + 3) + \left(\frac{5}{x}\right)(5^x + 3x)$

9) $(2x^5+5)(3x^2) + 60x^4(x^3-1)$

10) $\frac{(5x+2)(6x) - (3x^2)5}{(5x+2)^2}$

11) $\frac{(3^x+5^x)(\ln 2)2^x - 2^x((\ln 3)3^x + (\ln 5)5^x)}{(3^x+5^x)^2}$

12) $= \frac{4\ln(x)}{4\log_{12}x} = \frac{\ln x}{\log_{12}x}$, derivative $= \frac{(\log_{12}x)\frac{1}{x} - \ln x\left(\frac{1}{(\ln 12)x}\right)}{(\log_{12}x)^2}$

13) $5(3x+12)^4 \cdot 3$

14) $\frac{98(\ln x + 1)^{97}}{x}$

15) ke^{kx}

16) $= e^{-\log_5 x}$, derivative is $\left(e^{-\log_5 x}\right)\left(\frac{-1}{(\ln 5)x}\right)$

17) $14 \cdot H'(14x+2)$

18) $(15x^2 + e^x)\, H'(5x^3 + e^x)$

19) $(2x+3)\ln 5 = \ln 27$, $2x+3 = \frac{\ln 27}{\ln 5}$, $x = \frac{1}{2}\left(\frac{\ln 27}{\ln 5} - 3\right)$

20) $x = e^3$

21) $\left(\frac{8}{12}\right)^x = 3$, $x\ln\left(\frac{2}{3}\right) = \ln 3$, $x = \frac{\ln 3}{\ln(2/3)}$

22) $x = z \cdot w$

23) $x = \log_5 19$

24) $x = 0$

25) Minimize $\left(\frac{500}{s}\right)(.0025s^2 - .005s + 2.5) = s - 2.5 + \frac{1250}{s}$ Derivative is $1 - \frac{1250}{s^2}$

set equal to zero, find minimum when $s = \sqrt{1250}$ or about 35 mph