CHAPTER 11

A Portrait of Polynomials and Rational Functions

Section 11.1 A Portrait of Cubics from a Calculus Perspective

Problem 1.
Answers will vary; one possibility is: $f(x) = x(x+2)(x-3)$.

Problem 2.
Answers will vary; one possibility is: $f(x) = \frac{1}{4}(x+1)(x-2)^2$.

Problem 3.
Answers will vary; one possibility is: $f(x) = -(x-1)^3$.

Problem 4.
Answers will vary; two possibilities are: $f(x) = \frac{1}{3}x^3 - x^2$ and $f(x) = (x+1)(x-2)^2$.

Problem 5.
Answers will vary; one possibility is: $f(x) = (x-1)^3 = x^3 - 3x^2 + 3x - 1$.

Problem 6.
Answers will vary; one possibility is: $f(x) = x^3 + x$.

Problem 7.
Answers will vary; one possibility is: $f(x) = -\frac{2}{27}(x-3)^3$. As f is always decreasing, f must have exactly one zero. Hence, f is of the form $f(x) = k(x-3)^3$, where $k < 0$ is a constant. Using $f(0) = 2$, we have that $2 = k(0-3)^3$, from which we obtain $k = -\frac{2}{27}$.

Problem 12.

(a) $f(x) = x^3 - 3x + 1 \Rightarrow f'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x+1)(x-1)$. $0 = f'(x) \Rightarrow x = -1$ and $x = 1$ are the critical points. Now $f''(x) = 6x$ gives $f''(-1) = -6 < 0$ and $f''(1) = 6 > 0$. Therefore, $x = -1$ is a local maximum point and $x = 1$ is a local minimum point. Moreover neither of these critical points are absolute extreme points because $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

(b) $f(x) = x^3 + 3x + 1 \Rightarrow f'(x) = 3x^2 + 3 = 3(x^2+1)$. $0 = f'(x) \Rightarrow x^2 = -1$. Hence there are no real values of x for which $0 = f'(x)$ and consequently no critical points.

Problem 13.

(a) $f(x) = -x^3 - 3x^2 + 9x + 5 \Rightarrow f'(x) = -3x^2 - 6x + 9 = -3(x+3)(x-1)$. Now $0 = f'(x) \Rightarrow x = -3$ or $x = 1$. Now $f''(x) = -6x - 6$, and hence $f''(-3) = 12 > 0$ and $f''(1) = -12 < 0$. Thus there is a local minimum at $x = -3$ and a local maximum at $x = 1$. As $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$, neither of the local extrema are absolute extrema.

(b) $f(x) = x^3 + 3x^2 + 9x + 8 \Rightarrow f'(x) = 3x^2 + 6x + 9 = 3(x^2 + 2x + 3)$. Now $0 = f'(x) \Rightarrow x = \frac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2}$. Hence there are no real values of x for which $0 = f'(x)$ and consequently no critical points.

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Problem 2.
Answers will vary; one possibility: $P(x) = (x-1)(x+3)$.

Problem 3.
Answers will vary; one possibility: $P(x) = x(x+1)(x-5)$.

Problem 4.
Answers will vary; one possibility: $P(x) = x^4 + 5$.

Problem 5.
Answers will vary; one possibility: $P(x) = (x - \sqrt{2})^4$.

Problem 6.
Answers will vary; one possibility: $P(x) = x(x-9)^2(x-3)(x+e)$.

Problem 7.
Answers will vary; one possibility: $P(x) = -\dfrac{(x-\pi-1)^3}{(\pi+1)^3}$.

Problem 8.
Answers will vary; one possibility: $P(x) = (x+3)^4 + 2$.

Problem 9.
$P(x) = k(x-2)^2(x+3)^2$, for some $k \neq 0$. Now $-2 = P(0) = k(4)(9) = 36k \Rightarrow k = -\frac{1}{18}$. This answer is unique.

Problem 15.

(a) $f(x) = 2x^3 + 2x^2 - 12x = 2x(x+3)(x-2)$. The zeros are $x = -3, 0$, and 2.

(b) $g(x) = 2x^3 + 2x^2 + 12x = 2x(x^2 + x + 6)$. Now if $x^2 + x + 6 = 0$, then $x = \frac{-1\pm\sqrt{1-4(6)}}{2} = \frac{-1\pm\sqrt{-23}}{2}$, which are not real solutions. The only zero is $x = 0$.

Problem 17.

(a) $P(x) = x^3 - x^2 - 4x + 4 = (x^2-4)(x-1) = (x+2)(x-2)(x-1)$. The zeros are $x = -2, 1$, and 2.

(b) $Q(x) = x^3 - x^2 + 4x - 4 = (x-1)(x^2+4)$. Now $x^2 + 4 = 0$ has no real solutions; hence, the zero is $x = 1$.

Problem 22.

(a) Definitely false. A polynomial that is symmetric about the origin has odd degree.

(b) Definitely true. A polynomial that is symmetric about the origin has odd degree.

(c) Definitely false. A polynomial that is symmetric about the origin has odd degree, and odd-degree polynomials have at least one zero.

(d) Definitely true. A polynomial that is symmetric about the origin has odd degree, and odd-degree polynomials have at least one zero.

Problem 23.

(a) Definitely false. An even-degree polynomial can be symmetric about the y-axis but not the origin.

(b) Possibly true. True if $P(x) = x^4$; false if $f(x) = x^4 + x^3$.

(c) Definitely true. $\lim_{x\to-\infty} P(x) = \lim_{x\to\infty} P(x) = L$, where $L = \pm\infty$. As $P(0) = 0$, $P(x)$ must have at least one turning point.

(d) Definitely false. $\lim_{x\to-\infty} P(x) = \lim_{x\to\infty} P(x) = L$, where $L = \pm\infty$, which implies that $P(x)$ must have an odd number of turning points.

(e) Definitely true. $\lim_{x\to-\infty} P(x) = \lim_{x\to\infty} P(x) = L$, where $L = \pm\infty$, which implies that $P(x)$ must have an odd number of turning points.