Problem 2.

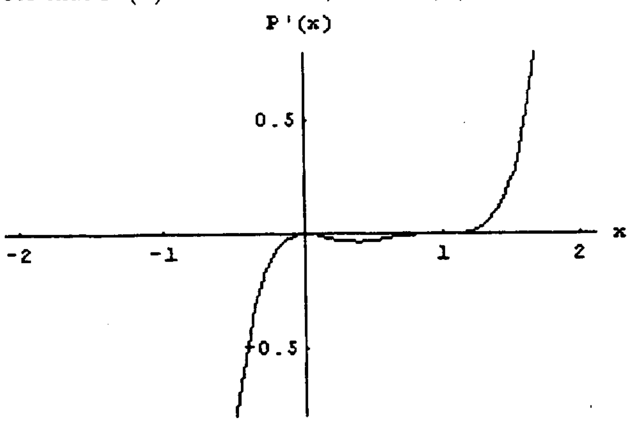
(a) Since $P'(x)$ is degree 5, P is degree 6.

(b) The critical points of P occur whenever $P'(x) = 0$, that is at $x = 0, -2$.

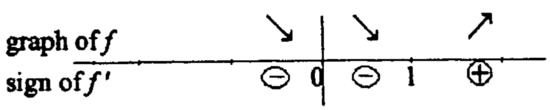
(c) Because $\lim_{x\to-\infty} P'(x) = -\infty$, $\lim_{x\to\infty} P'(x) = \infty$, and $P'(x)$ is continuous everywhere, $P(x)$ has an absolute minimum. We know it must be obtained at one of the critical points, so we calculate $P''(x) = 2x(x+2)^3 + 3x^2(x+2)^2$; unfortunately, $P''(0) = 0$ and $P''(-2) = 0$, so the second derivative test lends no information. (How could you have predicted that?) Instead we look at the sign of $P'(x)$; across $x = 0$ the sign remains positive, but across $x = -2$ the sign changes from negative to positive. Thus, there is an absolute minimum at $x = -2$. However, we cannot find the value of the function here since we don't have an expression for the function; we can determine only the value of the function relative to some other value of the function.

Problem 4.

(a) Note that $P'(x)$ has two zeros, at $x = 0, 1$; these are the critical points of $P(x)$.

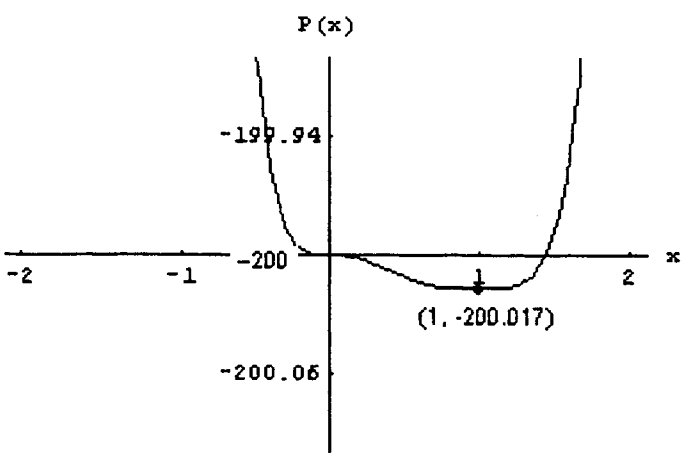


(b)



Note that $P' > 0$ on $(1, \infty)$ and $P' < 0$ on $(-\infty, 0) \cup (0, 1)$.

(c) Because $P(x) < 0$ at $x = 0$, $P(x) < 0$ at $x = 1$ since P is decreasing on the interval. Since $P'(x)$ is positive and increasing for $x > 1$, $P(x)$ must be increasing at an increasing rate for $x > 1$; because $P(x) < 0$ at $x = 1$, $P(x)$ must cross the x-axis.



Problem 5.

(a) False; (b) True; (c) Possible; (d) True; (e) False; (f) False; (g) True; (h) False.

Problem 9.

(a) The graph has the general shape of the graph of a fourth-degree polynomial function. There is a even order zero at $x = a$ and odd order zeros at $x = b$ and $x = c$. Hence the function has the form $P(x) = k(x-a)^2(x-b)(x-c)$, for some constant k. As the $y-$intercept is d, we have $d = P(0) = k(0-a)^2(0-b)(0-c) = ka^2bc \Rightarrow k = \frac{d}{a^2bc}$. Therefore, $P(x) = \frac{d}{a^2bc}(x-a)^2(x-b)(x-c)$

(b) The graph has the general shape of the graph of a sixth-degree polynomial function. There are a even order zeros at $x = a$ and $x = c$ and odd order zeros at $x = b$ and $x = 0$. Hence the function has the form $P(x) = kx(x-a)^2(x-b)(x-c)^2$, for some constant k. As the graph contains the point $(-1, 3)$, we have $3 = P(-1) = k(-1)(-1-a)^2(-1-b)(-1-c)^2 \Rightarrow k = \frac{3}{(a+1)^2(b+1)(c+1)^2}$. Therefore, $P(x) = \frac{3}{(a+1)^2(b+1)(c+1)^2}x(x-a)^2(x-b)(x-c)^2$.

Problem 1.

(a) The graph has a single simple zero at $x = 0$ and a vertical asymptote at $x = -1$. As the sign of y changes across the vertical asymptote, there is an odd power of $(x+1)$ in the denominator of the function. As there is a horizontal asymptote at $y = 2$, the degrees of the numerator and denominator of the function are equal, and the lead coefficient of the numerator is 2. Hence $y = \frac{2x}{x+1} = 2 - \frac{2}{x+1}$.

(b) The graph has simple zeros at $x = -2$ and $x = 0$ and a vertical asymptote at $x = -1$. As the sign of y does not change across the vertical asymptote, there is an even power of $(x+1)$ in the denominator of the function. As there is a horizontal asymptote at $y = 2$, the degrees of the numerator and denominator of the function are equal, and the lead the coefficient of the numerator is 2. Hence $y = \frac{2x(x+2)}{(x+1)^2} = -\frac{2}{(x+1)^2} + 2$.

(c) The graph has no zeros and always lies above the x–axis. There is a vertical asymptote at $x = -1$, which, in this case, implies that there is an even power of $(x+1)$ in the denominator of the function. As there is a horizontal asymptote at $y = 0$, the degree of the numerator of the function is less than the degree of the denominator of the function. Hence $y = \frac{2}{(x+1)^2}$.

(d) The graph has no zeros and vertical asymptotes at $x = -1$ and $x = 2$. The sign of y changes across both of these vertical asymptotes, which implies that there are odd powers of $(x+1)$ and $(x-2)$ in the denominator of the function. As there is a horizontal asymptote at $y = 0$, the degree of the numerator of the function is less than the degree of the denominator of the function. Hence the equation has the form $y = \frac{k}{(x+1)(x-2)}$, where k is a nonzero constant. As $y < 0$ for $|x| > 2$, $k < 0$. For simplicity, we choose $k = -1$. Therefore, $y = -\frac{1}{(x+1)(x-2)}$.

(e) The graph has no zeros and vertical asymptotes at $x = -1$ and $x = 2$. The sign of y changes across $x = -1$ but does not change across $x = 2$. Thus, there is an odd power $(x+1)$ and an even power of $(x-2)$ in the denominator of the function. As there is a horizontal asymptote at $y = 0$, the degree of the numerator of the function is less than the degree of the denominator of the function. Hence the equation has the form $y = \frac{k}{(x+1)(x-2)^2}$, where k is a nonzero constant. As $y < 0$ for $x > 2$, $k < 0$. For simplicity, we choose $k = -1$. Therefore, $y = -\frac{1}{(x+1)(x-2)^2}$

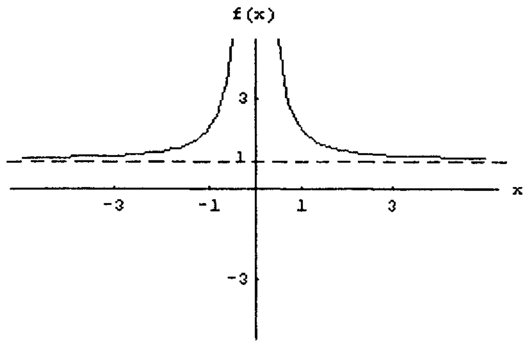
(f) The graph has no zeros, lies above the x–axis and has vertical asymptotes at $x = -3$ and $x = 1$. Thus, there are even powers of $(x+3)$ and $(x-1)$ in the denominator of the function. As there is a horizontal asymptote at $y = 1$, the equation has the form $y = \frac{1}{(x+3)^2(x-1)^2} + 1 = \frac{1+(x+3)^2(x-1)^2}{(x+3)^2(x-1)^2} = \frac{x^4+4x^3-2x^2-12x+10}{x^4+4x^3-2x^2-12x+9}$.

(g) The graph has no zeros, lies above the x–axis and has vertical asymptotes at $x = -3$ and $x = 1$. Thus, there are even powers of $(x+3)$ and $(x-1)$ in the denominator of the function. As there is a horizontal asymptote at $y = 0$, the degree of the numerator of the function is less than the degree of the denominator of the function. Hence the equation has the form $y = \frac{k}{(x+3)^2(x-1)^2}$. As $y > 0$ for $x > 1$, $k > 0$, and, for simplicity, we choose $k = 1$. Therefore, the equation is $y = \frac{1}{(x+3)^2(x-1)^2}$.

(h) The graph has no zeros and a vertical asymptote at $x = 0$. As the y changes sign across $x = 0$, there is an odd power x in the denominator of the function. As there are no vertical asymptotes, the degree of the numerator of the function is greater than the degree of the denominator. Hence the equation $y = \frac{x^2+1}{x}$ suffices.

Problem 4.

(a)



(b) (i) $\lim_{x\to\infty} f(x) = 1$;

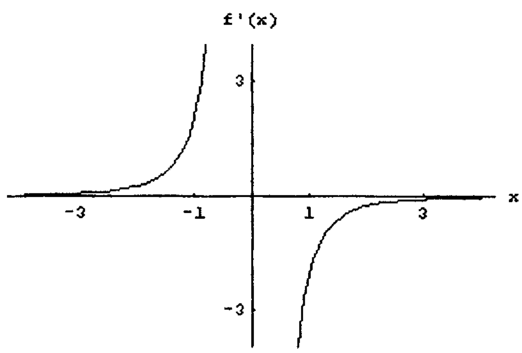
(ii) $\lim_{x\to 0} f(x) = \infty$;

(iii) $\lim_{x\to\infty} f'(x) = 0$;

(iv) $\lim_{x\to 0+} f'(x) = -\infty$;

(v) $\lim_{x\to 0^-} f'(x) = \infty$.

(c)

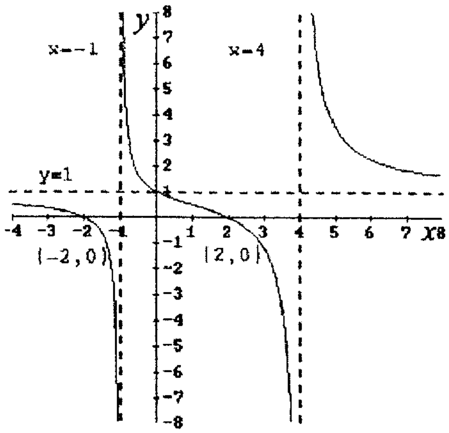


(d) $f(x) = 1 + \frac{1}{x^2} \Rightarrow f'(x) = -\frac{2}{x^3} \Rightarrow f''(x) = \frac{6}{x^4}$.

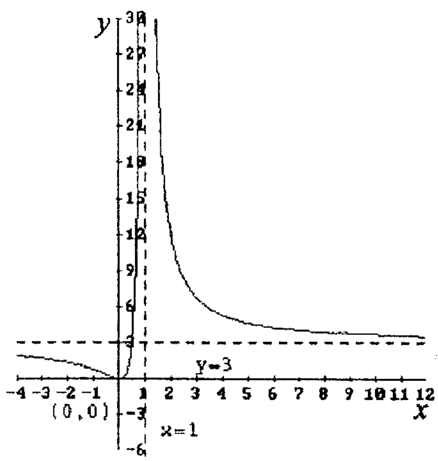
(e) Yes, the answers are consistent. This is a good place for the students to compare the graphical data with the analytic calculations.

Problem 7.

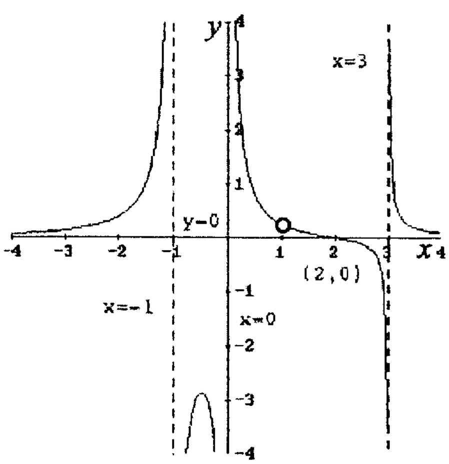
(a) Note that $f(x) = \frac{x^2-4}{x^2-3x-4} = \frac{(x+2)(x-2)}{(x+1)(x-4)}$. The x-intercepts are $x = \pm 2$; the y-intercept is $y = 1$; the vertical asymptotes are $x = -1, 4$; the horizontal asymptote is $y = 1$.



(b) The x-intercept is the origin; the y-intercept is the origin; the vertical asymptote is $x = 1$; the horizontal asymptote is $y = 3$.

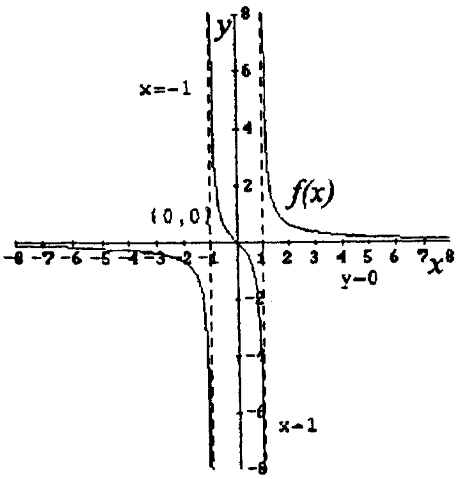


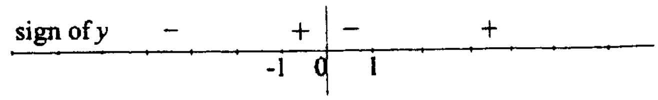
(c) The x-intercept is $x = 2$; there are no y-intercepts; the vertical asymptotes are $x = -1, 0, 3$; the horizontal asymptote is the $x-$axis; there is a hole at $\left(1, \frac{1}{3}\right)$.



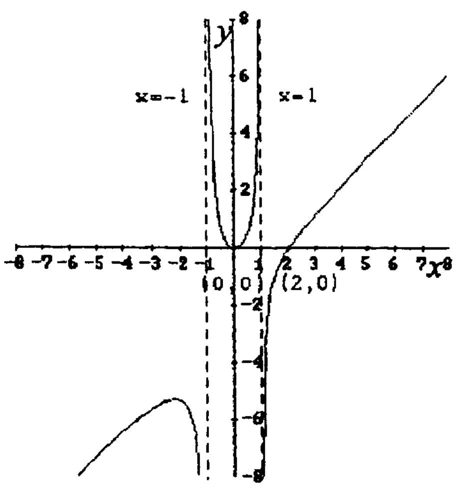
Problem 12.

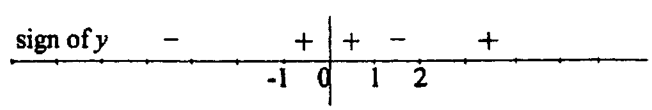
(i) The x-intercept is the origin; the y-intercept is the origin; the vertical asymptotes are $x = \pm 1$; the horizontal asymptote is the $x-$axis.



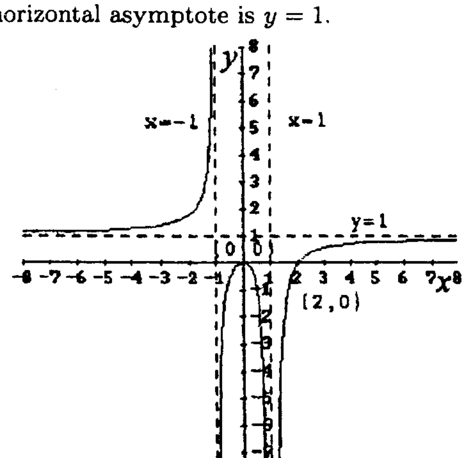


(ii),(iii) The x-intercepts are $x = 0, 2$; the y-intercept is the origin; the vertical asymptotes are $x = \pm 1$; there are no horizontal asymptotes.

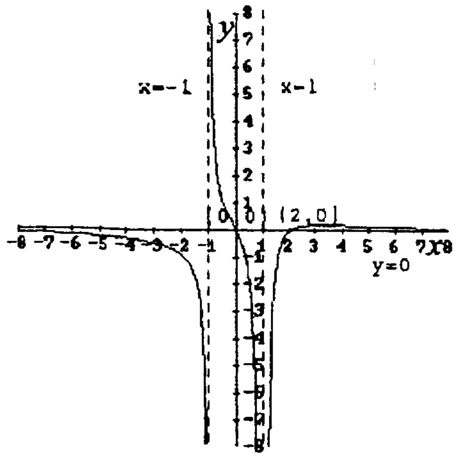


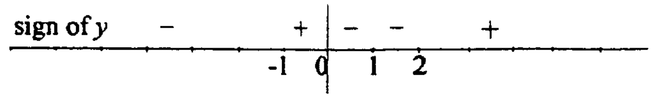


(iv) The x-intercepts are $x = 0, 2$; the y-intercept is the origin; the vertical asymptotes are $x = \pm 1$; the horizontal asymptote is $y = 1$.

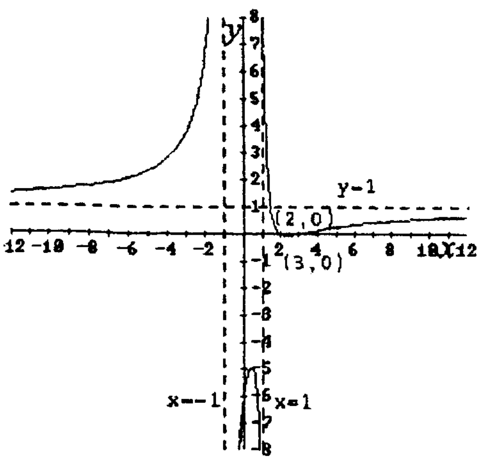


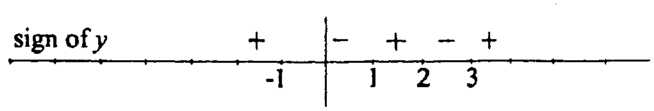
(v) The x-intercepts are $x = 0, 2$; the y-intercept is the origin; the vertical asymptotes are $x = \pm 1$; the horizontal asymptote is the $x-$axis.



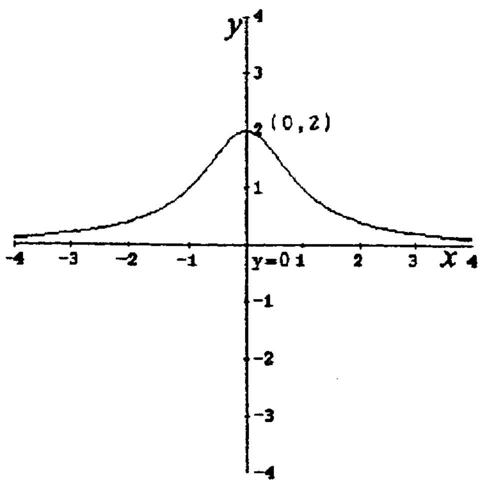


(vi) The x-intercepts are $x = 2, 3$; the y-intercept is $y = -6$; the vertical asymptotes are $x = \pm 1$; the horizontal asymptote is $y = 1$.





(vii) There are no x-intercepts; the y-intercept is $y = 2$; there are no vertical asymptotes; the horizontal asymptote is the x-axis. Note that $y < 0$ for all values of x.



(viii) The x-intercept is the origin; the y-intercept is the origin; there are no vertical asymptotes; the horizontal asymptote is $y = -1$. Note that $y < 0$ for all values of x.