Math Xa
Worksheet—Concavity and the Second Derivative
Solutions

Fall 2003

1. Let $f(x) = e^x/x$.

   (a) Find all the critical points of $f(x)$.

   Solution. Since
   $$f'(x) = \frac{xe^x - e^x}{x^2},$$
   can only be zero when $xe^x - e^x = e^x(x-1) = 0$, we have a critical point at $x = 1$. Even though $f'(x)$ is not defined at $x = 0$, this point is not a critical point since the original function is not defined at $x = 0$.

   (b) Identify all local extrema.

   Solution. We will use the Second Derivative Test. Since
   $$f''(x) = \frac{2e^x - 2xe^x + x^2e^x}{x^3},$$
   we know that $f''(1) = e > 0$, which tells us that $x = 1$ is a local minimum.

   (c) Does f have an absolute maximum value? If so, where is it attained? What is its value?

   Solution. The function f does not have an absolute maximum, since
   $$\lim_{x\to\infty} f(x) = \infty.$$

(d) Does f have an absolute minimum value? If so, where is it attained? What is its value?

Solution. Similarly, the function f does not have an absolute minimum, since

$$\lim_{x\to 0^-} f(x) = -\infty.$$

(e) Answer parts (c) and (d) if x is restricted to $(0, \infty)$.

Solution. Since f is continuous on $(0, \infty)$ and

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \infty \\ \lim_{x\to 0^+} f(x) &= \infty, \end{aligned}$$

there is no absolute maximum. However, f does have an absolute minimum at $x = 1$ of e, since the only turning point of f occurs at $x = 1$.

2. Find all local extrema of $f(x) = \frac{1}{5}x^5 - x^4 + \frac{4}{3}x^3 + 2$. Also determine the concavity and inflection points of f.

Solution. First,

$$f'(x) = x^4 - 4x^3 + 4x^2 = x^2(x-2)^2$$
$$f''(x) = 4x^3 - 12x^2 + 8x = 4x(x-1)(x-2).$$

Thus, the critical points of f are at $x = 0$ and $x = 2$. Since $f''(0) = 0$ and $f''(2) = 0$, we cannot apply the Second Derivative test. However, $f'(x) \geq 0$ for all x. Therefore, the First Derivative Test tells us that f has no local extrema.

To find the inflection points for f, first notice that $f''(x) = 0$ when $x = 0$, 1, or 2. Since $f''(x)$ is positive (concave up) on the intervals $(0, 1)$ and $(2, \infty)$ and negative (concave down) on $(-\infty, 0)$ and $(1, 2)$, the inflection points of f occur at $x = 0$, 1, and 2.

3. Suppose that f is a continuous function such that $f(3) = 2$, $f'(3) = 0$, and $f''(3) = 3$. At $x = 3$, does f have a local maximum, a local minimum, neither a local maximum nor a local minimum, or is it impossible to determine? Explain your answer.

Solution. The Second Derivative Test guarantees that f has a local minimum at $x = 3$, because $f'(3) = 0$ and $f''(3) > 0$.