Name: Solutions

Section Leader: circle one

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Mathematics Xa

Final Examination

January 18, 1995

Please read all the problems carefully. Problems # 3 and 5 are multiple choice. For all other problems you must show all your work and explain your reasoning clearly in order to get credit. Calculators are permitted, but you must explain your reasoning; your reasoning cannot simply be "my calculator says so".

You will be given three hours to do this examination. Please use your time wisely.

Think clearly and do well!

Question	Points	Score
1	13	
2	9	
3	10	
4	10	
5	6	
6	15	
7	6	
8	7	
9	12	
10	12	
Total	100	

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1. (13 points) Find the derivative, f ', of each of the following functions: For problems a) - c) please simplify your answers. You need ***not*** simplify your answers in parts d) - f).

a) $f(x) = \ln x^3$

$$\frac{1}{x^3} \cdot 3x^2 = \frac{3}{x}$$

b) $f(x) = \frac{5x}{e^{2x}}$

$$\frac{e^{2x} \cdot 5 - 5x \cdot e^{2x} \cdot 2}{e^{4x}} = \frac{5-10x}{e^{2x}}$$

c) $f(t) = \ln\left(\frac{t+1}{2t}\right)$

$$\frac{1}{\left(\frac{t+1}{2t}\right)} \cdot \frac{2t-(t+1)2}{4t^2} = \frac{2t}{(t+1)} \cdot \left(\frac{2t-2t-2}{4t^2}\right) = \frac{-4t}{4t^2(t+1)} = \frac{-1}{t(t+1)}$$

d) $f(x) = 3^t t^\pi$

$$(\ln 3)\, 3^x \cdot t^\pi + 3^t \pi t^{\pi-1}$$

e) $f(x) = (\ln x)^3 + \left(\frac{x}{2}\right)^3 + \frac{2}{x^3}$

$$3(\ln x)^2 \cdot \frac{1}{x} + \frac{3}{2}\left(\frac{x}{2}\right)^2 - 6x^{-4}$$

f) $f(t) = \sqrt{\ln(3x^2 + x)}$

$$\frac{1}{2}\left[\ln(3x^2+x)\right]^{-\frac{1}{2}} \cdot \frac{1}{(3x^2+x)} \cdot (6x+1)$$

2. (9 points) At time $t = 0$ a ball is thrown. Its height (measured in feet above the ground) at time t (t in seconds) is given by

$$h(t) = -16t^2 + 40t + 6.$$

Please give your answers to the following questions rounded off to two decimal places.

a) How many seconds after it is thrown does the ball reach its maximum height?

$$h'(t) = -32t + 40 = 0 \implies t = \frac{40}{32} = 1.25$$

b) How many seconds after it is thrown does the ball hit the ground?

$-16t^2 + 40t + 6 = 0$ quadratic eq:

$$\frac{-40 \pm \sqrt{40^2 + 4 \cdot 16 \cdot 6}}{-32} = \frac{-40 \pm 44.54}{32} = 2.64$$

c) From what height was the ball thrown?

$h(0) = 6$ ft

d) What was the ball's initial velocity?

$h'(0) = 40$

e) At what value of t (if any) is the speed of the ball the same as the speed of the ball at the time it was thrown? (Here we are asking about speed as opposed to velocity.)

$$h'(t) = -32t + 40 = -40$$

$$t = 2.5$$

3. (10 points) Some functions are listed below. a,b,c, are **positive constants**. Each function corresponds to a set of curves. For each function, choose one graph from those given below that could be the graph of the function given:

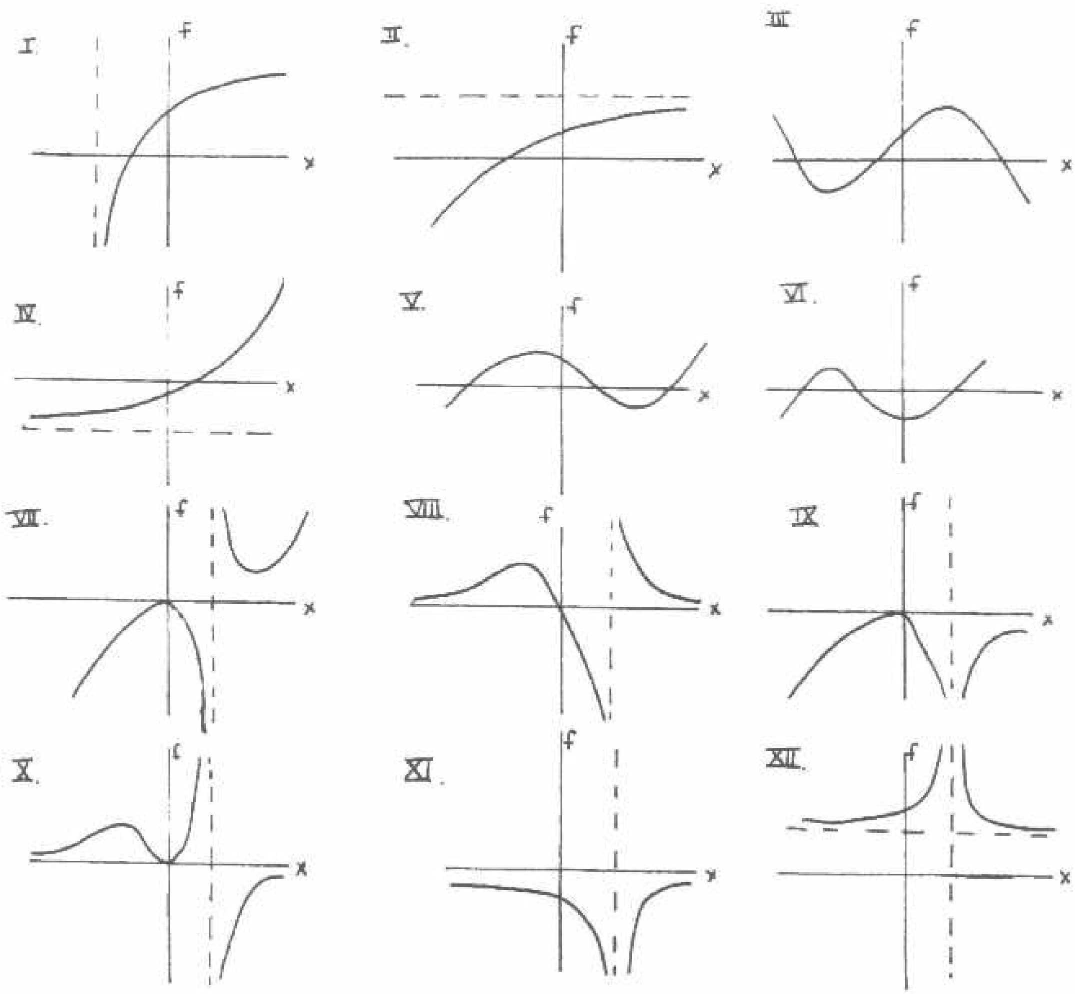
a) $f(x) = \ln(x+a)$ I

b) $f(x) = a^x - b$ IV

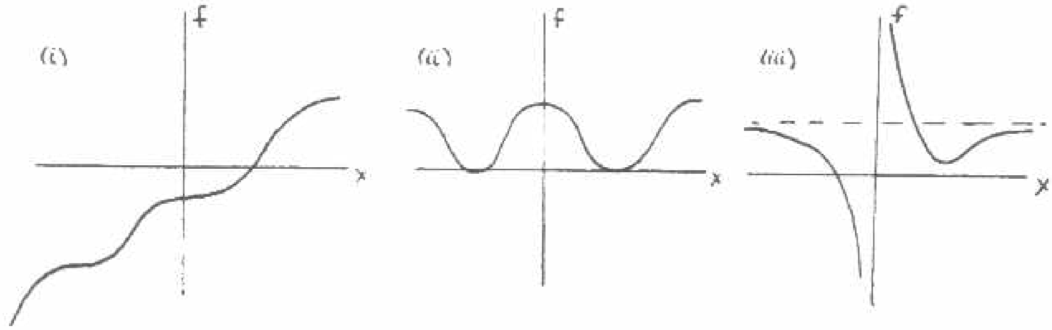
c) $f(x) = (x - a)(x - b)(x + c)$ V

d) $f(x) = \frac{-a}{(x-b)^2}$ XI

e) $f(x) = \frac{x^2}{x-b}$ VII



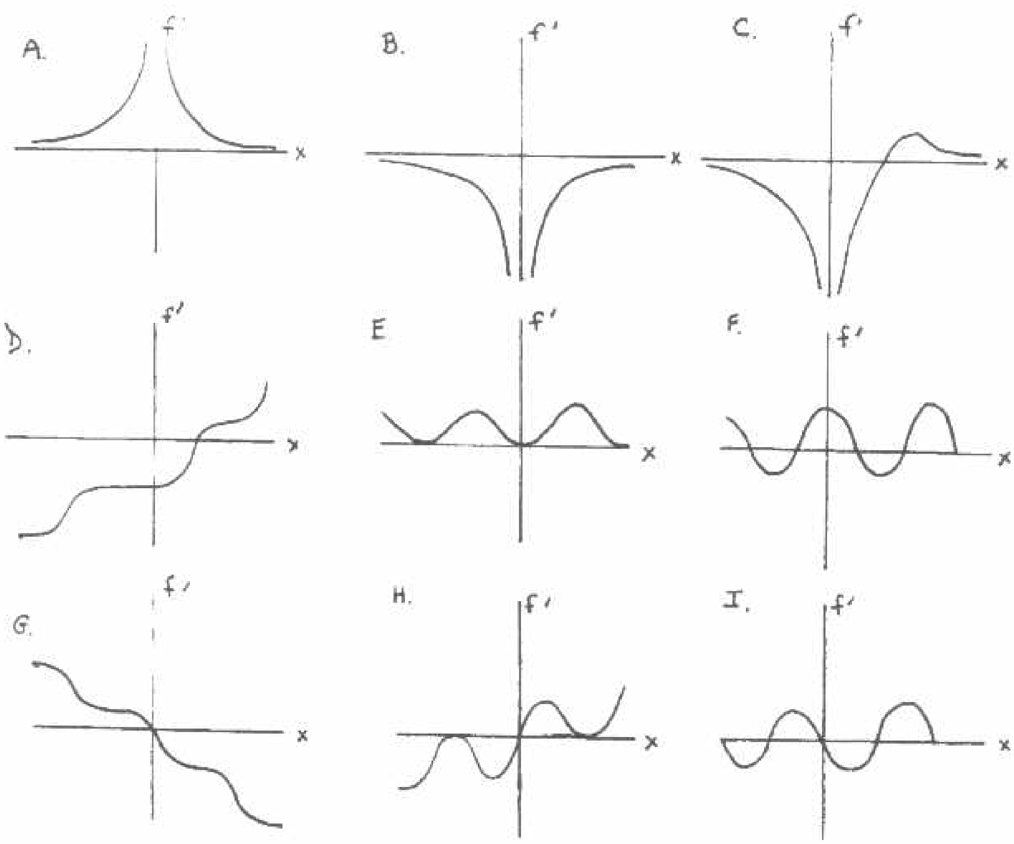
5. (6 points) To each of the graphs of f given below, match the graph of its derivative f '.



If the graph of f is given by (i), f ' could be given by E

If the graph of f is given by (ii), f ' could be given by I

If the graph of f is given by (iii), f ' could be given by C



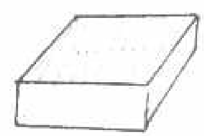
6. (15 points) A grocer wants to package strawberries in cardboard boxes with a **square** base. The lid (top) of each box is plastic (instead of cardboard) so that the customer can see the produce inside. Each box is to have a fixed volume of V cubic feet. Assume that plastic costs 1 cent per square foot and cardboard costs 2 cents per square foot.

a) How long should the sides of the base be in order to minimize the total cost of the strawberry box?) Your answer will be in terms of V - i.e. it will have V in it.)

Note: Please explain how you know that the length you found actually *minimizes* cost.

$V = l \cdot w \cdot h = w^2 \cdot h$

Minimize $C = w^2 + 2w^2 + 4wh$



$h = \frac{V}{w^2} \Rightarrow C = w^2 + 2w^2 + 4w \cdot \frac{V}{w^2}$

minimize:

$C = 3w^2 + \frac{4V}{w}$ $\quad C'(w) = 6w - \frac{4V}{w^2} = 0$ so

$6w^3 = 4V \Rightarrow \boxed{w^* = \left(\frac{2}{3}V\right)^{1/3}}$

$C''(w^*) = 6 + \frac{8V}{(w^*)^3} = 6 + \frac{8V}{\left(\frac{2}{3}V\right)} = 6 + \frac{3}{2} \cdot 8 > 0$

So this is a min.

b) What is the ratio of the height of the box to the length of the base if cost is to be minimized? Please simplify your answer.

$$\frac{h}{w} = \frac{\frac{V}{w^2}}{w} = \frac{V}{w^3} = \frac{V}{2/3\, V} = \boxed{\frac{3}{2}}$$

7. (6 points) Solve the following equations for the variable indicated:

a) $A \cdot B^{kt} = (HC)^{p+t}$ (t)

$$\ln A + Kt\ln(B) = (p+t)\ln(HC)$$

$$t \cdot K\ln(B) = t \cdot \ln(HC) + p\ln(HC) - \ln A$$

$$t(K\ln(B) - \ln(HC)) = p\ln(HC) - \ln A \quad t = \frac{p\ln HC - \ln A}{K\ln(B) - \ln(HC)}$$

b) $4\ln(2x) + 3 = 6\ln(7x) + 1$ (x)

$$e^{(4\ln(2x)+3)} = e^{(6\ln(7x)+1)}$$

$$(e^{\ln 2x})^4 \cdot e^3 = (e^{\ln 7x})^6 \cdot e$$

$$(2x)^4 \cdot e^3 = (7x)^6 \cdot e$$

$$\frac{2^4 \cdot e^3}{7^6 e} = \frac{x^6}{x^4}$$

$$\frac{2^4}{7^6} \cdot e^2 = x^2$$

$$\boxed{x = \frac{4}{7^3} \cdot e}$$

8. (7 points) Using the limit definition of derivative, find $f'(x)$ if $f(x) = (2x - 3)^{-1}$. When you have finished, tell us whether or not you have gotten the correct answer.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{2(x+h)-3} - \frac{1}{2x-3}}{h} =$$

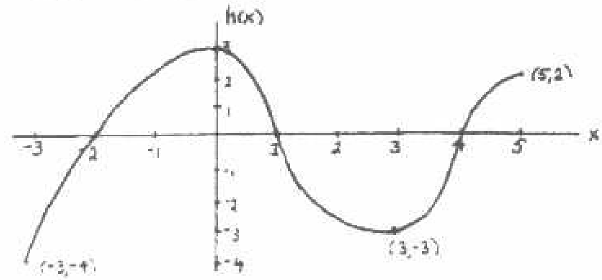
$$\lim_{h \to 0} \frac{2x - 3 - (\cancel{2x} + 2h - \cancel{3})}{h(2x+2h-3)(2x-3)} = \lim_{h \to 0} \frac{-2h}{h(2x+2h-3)(2x-3)} =$$

$$\frac{-2}{(2x-3)^2}$$

using chain rule, $f'(x) = -(2x-3)^{-2} \cdot 2 = \frac{-2}{(2x-3)^2}$

so this is correct.

9. (12 points) The graph of h(x) is given below.



Please read the questions carefully to be sure that you are answering what is asked.

I. For which values of x is h'(x) negative? Indicate your answer by shading in the region(s) on the number line below:

-3 -2 -1 0 1 2 3 4 5

II. This part of the question concerns the function f given by $f(x) = [h(x)]^2$ where the graph of h(x) is given above.

a) Find all the points (x values) at which the graph of f(x) has a horizontal tangent line.

$f'(x) = 2 \cdot h(x) \cdot h'(x)$ so $f' = 0 \iff h' = 0$ or $h = 0$

so $x = -2, 0, 1, 3, 4, 5$

b) On the number line below, indicate the sign of f '(x). $f' > 0 \iff h, h'$ have same sign
$f' < 0 \iff h, h'$ have diff. signs

sign of f '   − + + − + + − +

-3 -2 -1 0 1 2 3 4 5

c) Identify the x-coordinates of all the local maxima and minima of f..

x-coords. of local maxima: 0, 5

x-coords. of local minima: 3

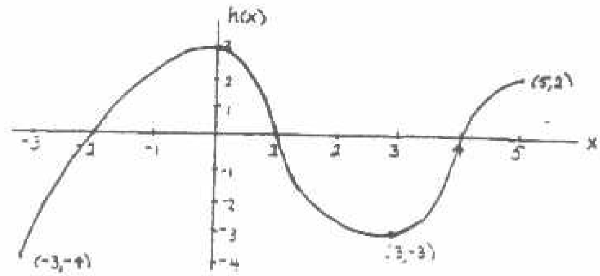
d) On the interval [-3,5], what is the largest value taken on by f?
(I.e. what is the global or absolute maximum value of f for x in [-3,5]?)

Largest value of f: 3

e) On the interval [-3,5], what is the smallest value taken on by f?

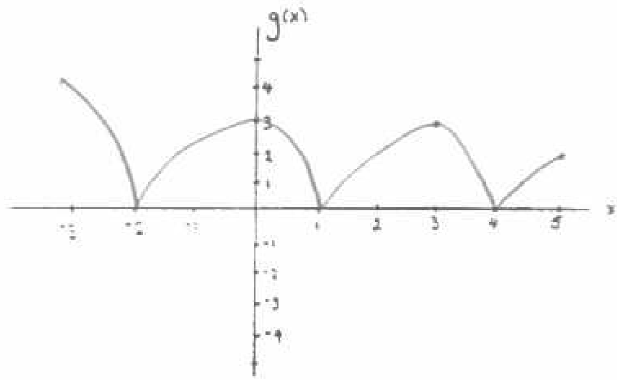
Smallest value of f: -4

III. Here is the graph of h(x).



Let g(x) = |h(x)|, the absolute value of h(x).

a) Sketch the graph of g(x).



b) Is g'(x) defined everywhere? If not, where is g'(x) undefined?

g'(x) undefined at x=-2, 1, 4 b/c there is no tangent line at these points.

10. (12 points) In a certain area, the population of rabbits doubles every 10 years. The rabbit population was 100 in 1970. In 1970 the squirrel population was 600 and since then it has been decreasing at a rate of 5% every 2 years.

a. When are the two populations equal? Please give the date - including the year and the month. At that time, what will the populations be?

rabbits: $R(t) = R_0 \cdot (2^{1/10})^t = 100 \cdot (1.07)^t$

squirrels: $S(t) = S_0 \cdot (.95)^{t/2} = 600 \cdot (.97)^t$

$100(1.07)^t = 600(.97)^t$

$\ln 100 + t\ln(1.07) = \ln 600 + t\ln(.97)$

$t = \dfrac{\ln 600 - \ln 100}{\ln 1.07 - \ln .97} = 18.26$ yrs

1988, April

pop = 344

b. What is the instantaneous rate of change of rabbit population when the rabbit population is 350?

$R'(t) = 100 \cdot \ln 1.07 \cdot (1.07)^t$. When $R = 350$,

$350 = 100 \cdot (1.07)^t \quad \ln 350 = \ln 100 + t\ln 1.07$

$t = 18.52$. So $R'(t) = 23.69$ rabbits/yr

c. When will the instantaneous rate of change of the rabbit population be 50 rabbits/year?

$50 = 100 \cdot \ln 1.07 (1.07)^t$

$\ln 50 = \ln 6.77 + t\ln(1.07)$

$t = 30$ yrs