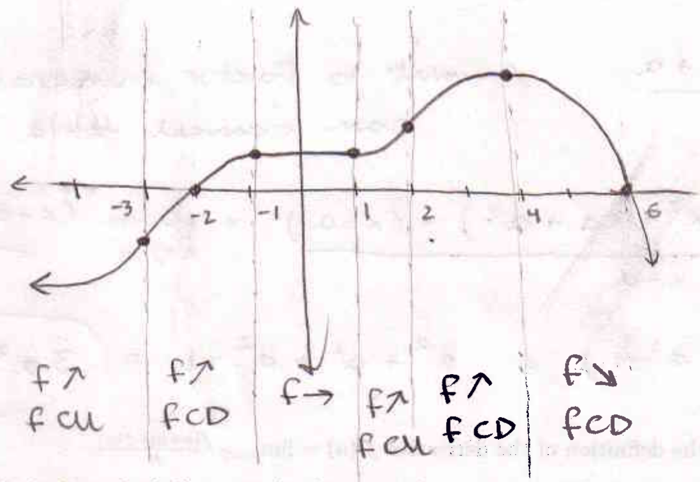
Math Xa Fall 2003 Name SOLUTIONS

Lab 5: Derivatives Section Instructor __________

October 28, 2003 Collaborators __________

1. Sketch the graph of a single function f that satisfies all of the following conditions.

   (a) $f(x) > 0$ for x in $(-2, 6)$
   (b) $f(x) < 0$ for x in $(-\infty, -2) \cup (6, \infty)$
   (c) $f(x) = 0$ for $x = -2, 6$
   (d) $f'(x) > 0$ for x in $(-\infty, -1) \cup (1, 4)$
   (e) $f'(x) < 0$ for x in $(4, \infty)$
   (f) $f'(x) = 0$ for x in $[-1, 1]$ and for $x = 4$
   (g) $f''(x) > 0$ for x in $(-\infty, -3) \cup (1, 2)$
   (h) $f''(x) < 0$ for x in $(-3, -1) \cup (2, \infty)$



Key

f ↗ = "f increases"

f ↘ = "f decreases"

f CU = "f concave up"

f CD = "f concave down"

2. The height in feet of a ball t seconds after it is thrown is given by $h(t) = -16t^2 + 32t + 48$.

   (a) Find the time t_m at which the ball reaches its maximum height.

$a = -16 \quad b = 32 \quad c = 48$

$0 = v(t) = 2at + b = 2(-16)t + 32 = -32t + 32$

$0 = -32t_m + 32$

$32t_m = 32$

$t_m = 1\,s$

   (b) Find the average velocity of the ball over the time interval $[0, t_m]$.

$\text{avg. velocity} = \dfrac{\Delta\ \text{height}}{\Delta\ \text{time}} = \dfrac{h(t_m) - h(0)}{t_m - 0}$

$= \dfrac{h(1) - 48}{1} = -16 + 32 + 48 - 48 = 16\ \text{ft}/\text{s}$

   (c) At what t value in the interval $[0, t_m]$ does the ball have an instantaneous velocity equal to the average velocity you found in part (c)?

For what t does $v(t) = 16$?

$16 = v(t) = -32t + 32$

$32t = 16$

$t = 1/2\ s$

3. If the tangent line to the curve $y = f(x)$ at the point $(5, 1)$ also passes through the point $(2, 3)$, then what is $f'(5)$?

$f'(5)$ = slope of tangent line at 5

$= \frac{\Delta y}{\Delta x} = \frac{3-1}{2-5} = \frac{2}{-3} = -2/3$

4. Let $f(x) = x^3 - x + 1$.

(a) Find $f'(a)$ using the definition of the derivative $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^3 - x + 1 - (a^3 - a + 1)}{x - a}$$

$$= \lim_{x \to a} \frac{x^3 - a^3 - x + a}{x - a}$$

(want to factor numerator so we can cancel $x - a$)

$$= \lim_{x \to a} \frac{(x-a)(x^2 + xa + a^2) - (x-a)}{x-a} = \lim_{x \to a} \frac{(x-a)(x^2 + xa + a^2 - 1)}{x-a}$$

$$= \lim_{x \to a} (x^2 + xa + a^2 - 1) = a^2 + a^2 + a^2 - 1 = 3a^2 - 1$$

(b) Find $f'(a)$ using the definition of the derivative $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(a+h)^3 - (a+h) + 1 - (a^3 - a + 1)}{h}$$

$$= \lim_{h \to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a - h + 1 - a^3 + a - 1}{h}$$

$$= \lim_{h \to 0} \frac{3a^2h + 3ah^2 + h^3 - h}{h} = \lim_{h \to 0} 3a^2 + 3ah + h^2 - 1$$

$$= 3a^2 - 1$$

(c) Which of the two methods do you prefer and why? Please answer in complete sentences.

The first method requires me to remember how to factor $x^3 - a^3$. The second method requires me to expand $(a+h)^3$. Since I find multiplying easier than factoring, I prefer the second method.

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