Math Xa Fall 2003 Name SOLUTIONS

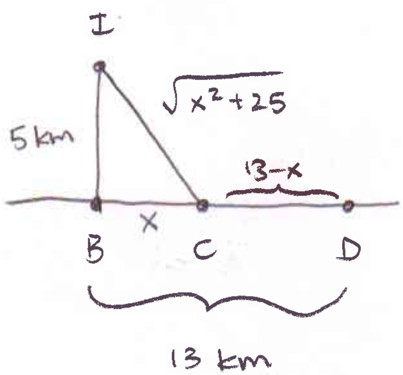
Lab 10: Optimization Section Instructor

December 11, 2003 Collaborators

Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than land because air generally rises over land and falls over water during the day.

A bird with these tendencies is released from an island that is 5 km from the nearest point B on a straight shoreline, flies to a point C on the shoreline, and then flies along the shoreline to its nesting area D. Assume that the bird instinctively chooses a path that will minimize its energy expenditure. Points B and D are 13 km apart.

1. In general, if it takes 1.4 times as much energy to fly over water as land, to what point C should the bird fly in order to minimize the total energy expended in returning to its nesting area?



Domain = $[0,13]$

$$E(x) = 1.4\sqrt{x^2+25} + 13 - x$$

$$= 1.4(x^2+25)^{1/2} + 13 - x$$

$$E'(x) = 1.4\left(\tfrac{1}{2}\right)(x^2+25)^{-1/2}(2x) - 1$$

$$= \frac{1.4x}{\sqrt{x^2+25}} - 1 = \frac{1.4x - \sqrt{x^2+25}}{\sqrt{x^2+25}}$$

$$E'(x) = 0 \iff 1.4x - \sqrt{x^2+25} = 0$$

$$1.4x = \sqrt{x^2+25}$$

$$1.96x^2 = x^2 + 25$$

$$0.96x^2 = 25$$

$$x = \sqrt{\frac{25}{0.96}} \approx 5.103 \text{ km}$$

critical point

$$E(0) = 1.4 \cdot 5 + 13 = 20$$

$$E(5.103) \approx 17.90$$ ← minimum

$$E(13) \approx 19.50$$

$\Rightarrow$ The bird should fly to a point C 5.103 km down the shore from B!

2. Let W and L denote the energy expended (in joules) per kilometer flown over water and land, respectively.

(a) What would a large value of the ratio $\frac{W}{L}$ mean in terms of the bird's flight? What would a small value mean?

large $\frac{W}{L}$ means more energy needed over water $\Rightarrow$ C should be closer to B

small $\frac{W}{L}$ means less energy needed over water $\Rightarrow$ C should be closer to D

(b) To what point C should the bird fly in order to minimize the total energy expended in returning to its nesting area? (Your answer should be in terms of the ratio $\frac{W}{L}$.)

$$E(x) = \frac{W}{L}\sqrt{x^2+25} + 13 - x$$

$$E'(x) = \frac{\frac{W}{L}x - \sqrt{x^2+25}}{\sqrt{x^2+25}} = 0 \Leftrightarrow \frac{W}{L}x - \sqrt{x^2+25} = 0$$

$$\frac{W}{L}x = \sqrt{x^2+25}$$

$$\left(\frac{W}{L}\right)^2 x^2 = x^2 + 25$$

$$x^2\left(\left(\frac{W}{L}\right)^2 - 1\right) = 25$$

$$x = \sqrt{\frac{25}{\left(\frac{W}{L}\right)^2 - 1}}$$

$$\boxed{\frac{5}{\sqrt{\left(\frac{W}{L}\right)^2 - 1}}}$$ km down the shore from B

3. If the ornithologists observe that birds of a certain species reach the shore at a point 4 km from B, how many times more energy does it take a bird to fly over water than land?

$$4 = \frac{5}{\sqrt{\left(\frac{W}{L}\right)^2 - 1}}$$

$$\sqrt{\left(\frac{W}{L}\right)^2 - 1} = 5/4$$

$$\left(\frac{W}{L}\right)^2 - 1 = 25/16$$

$$\left(\frac{W}{L}\right)^2 = 41/16$$

$$\frac{W}{L} = \frac{\sqrt{41}}{4} \approx \boxed{1.601 \text{ times}}$$