Name: Solutions

Instructor: _____

Circle one: MWF 10 MWF 11

MATHEMATICS Xa

First Examination

November 2, 1994

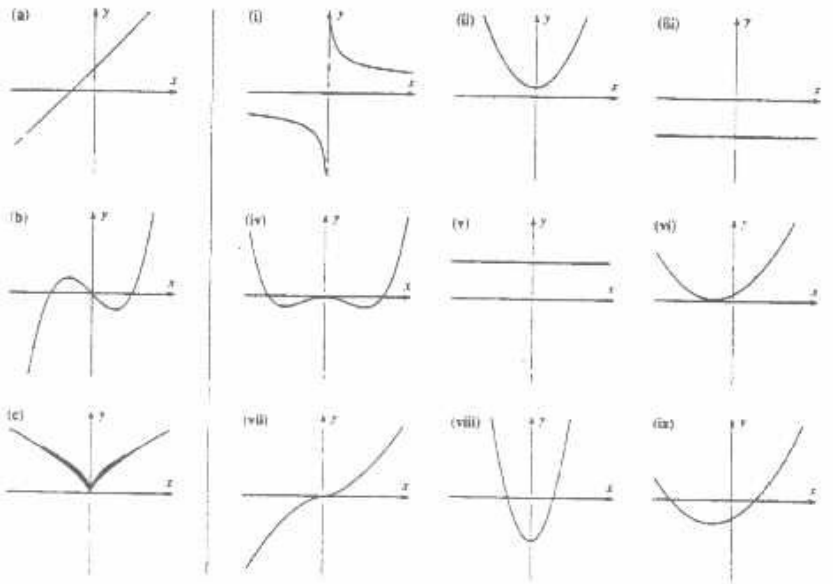
Problems # 1 - 3 are short answer: answers require no explanation. In problems # 4 - 11 you must show all your work on the exam paper and explain your reasoning clearly. Little credit will be given for unsubstantiated answers.

You can freely use the result that if $f(x) = ax^2 + bx + c$ then $f'(x) = 2ax + b$.

Please think clearly and do well!

Question	Points	Score
1	6	
2	6	
3	10	
4	11	
5	10	
6	8	
7	10	
8	9	
9	10	
10	10	
11	10	
Total	100	

1. (6 points) For each of the functions graphed in (a) - (c), determine which of the graphs i - ix could be the graph of its derivative.



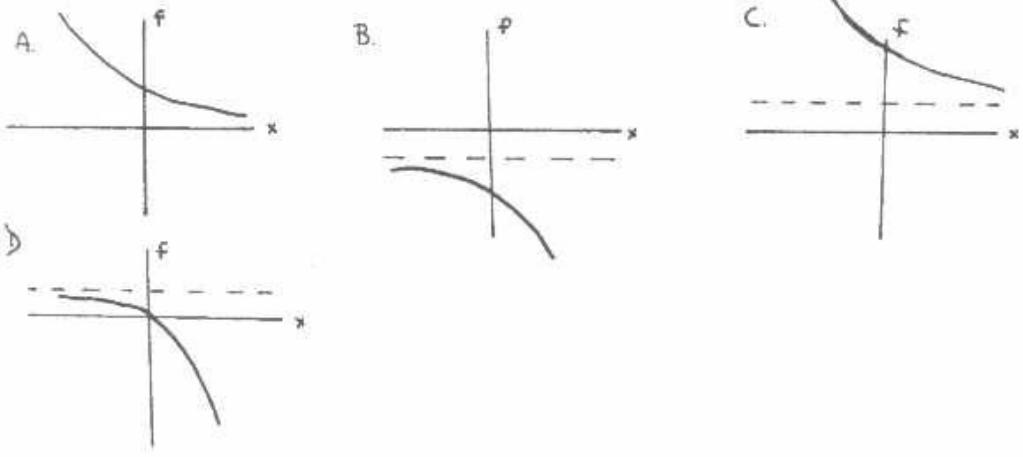
f	f'
a	V
b	VIII
c	I

2. (6 points) Let a be a number greater than 1 and b be a number between 0 and 1 : $a>1$; $0<b<1$.
To each of the three functions below, match the appropriate graph (A - D) of f.

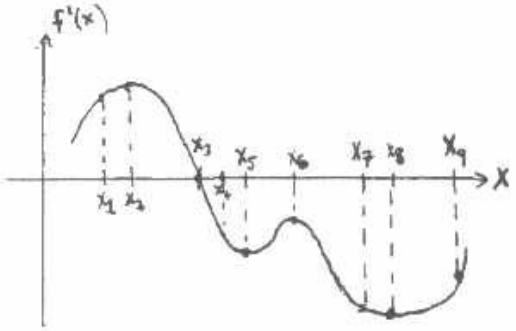
i) $f(x) = a^{-x} + b$ corresponds to graph C

ii) $f(x) = ab^{x}$ corresponds to graph A

iii) $f(x) = -a^{3x} + 1$ corresponds to graph D



3. (10 points) Below is a graph of f ' (***NOT*** of f !)



Circle the correct answer.

a) Choose the point listed at which f(x) is least: x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 (**x_9**)

b) Choose the point listed at which f(x) is greatest: x_1 x_2 (**x_3**) x_4 x_5 x_6 x_7 x_8 x_9

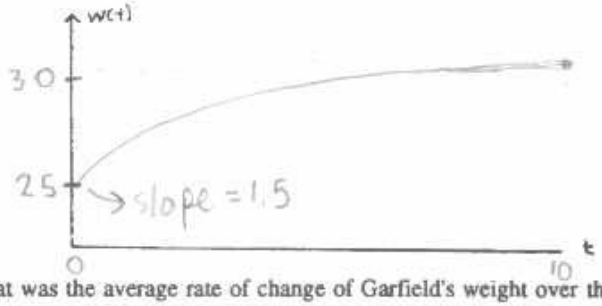
c) Choose the point listed at which f '(x) is least: x_1 x_2 x_3 x_4 x_5 x_6 x_7 (**x_8**) x_9

d) Choose the point listed at which f '(x) is greatest: x_1 (**x_2**) x_3 x_4 x_5 x_6 x_7 x_8 x_9

e) Over what interval(s) is the graph of f decreasing and concave down? $[x_3, x_5]$, $[x_6, x_8]$

4 (11 points) The comic strip shown below says that recently Garfield has "slowed [his] weight gain to a crawl." Assume this means that over the past ten days (time t = 0 to t = 10) his weight has been increasing at a decreasing rate. Suppose that at t = 0 Garfield's weight was 25 pounds and by t = 10 he tipped the scales at 30 pounds. Assume that at t = 0 he was gaining weight at an instantaneous rate of 1.5 pounds per day.

a) On the axes provided, sketch a graph of w(t), Garfield's weight plotted against time (t in days). Label any points and slopes you know.



b) What was the average rate of change of Garfield's weight over the interval [0,10]?

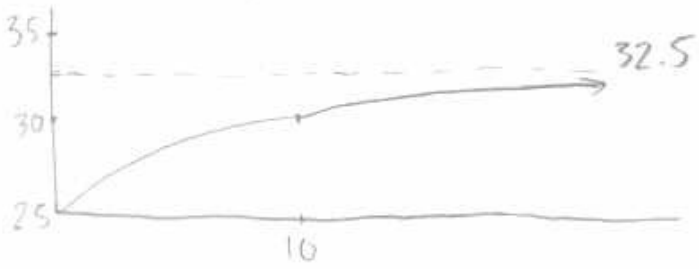
$$\frac{30-25}{10-0} = \frac{5}{10} = \frac{1}{2}\ \text{lb/day}$$

c) Find an upper bound and a lower bound for Garfield's weight at t = 4. Your upper and lower bounds should differ by no more than two pounds. Explain clearly how you know that Garfield's weight at t = 3 is less than your upper bound and greater than your lower bound.

30 is clearly an upper bound at $t=4$. $\frac{1}{2}$ is the average rate of change, so $25+\frac{1}{2}\cdot 4 = 27$ is a lower bound. Since the function is concave, the average of these 2 bounds, 28.5, is also a lower bound.

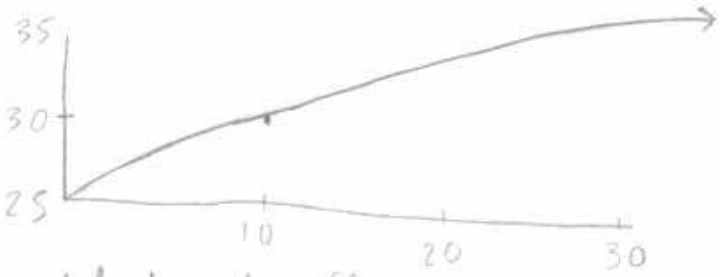
d) Assuming that Garfield continues to gain weight at a decreasing rate,is it possible that his weight will always remain below 35 pounds? Is it possible that his weight will surpass 35 pounds? Please explain your answer using a sketch.

Both are possible.



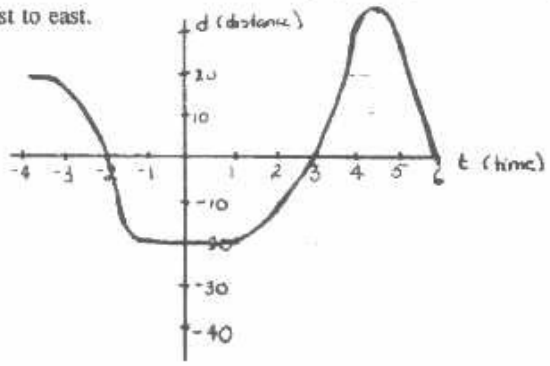
could have asymptote at, say, 32.5 lb

Or

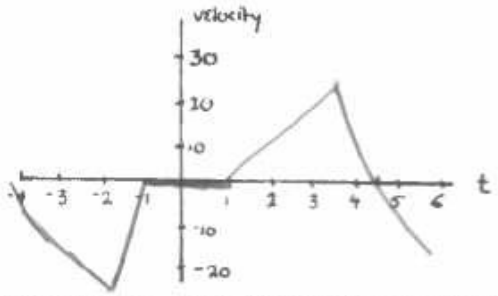


could level off very gradually

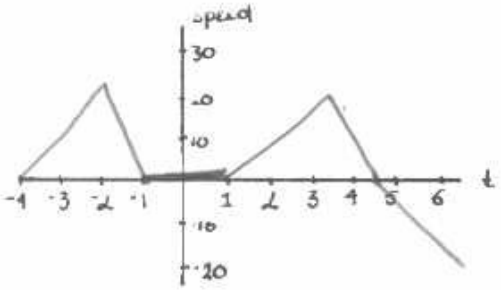
5. (10 points) The following is a graph of a trip, with our usual conventions. The benchmark time is noon and the benchmark location is Oklahoma City. We are on Route 66. East of Oklahoma City is denoted by a positive distance, west of Oklahoma City by a negative. Velocity is positive when we are travelling from west to east.



a) Sketch a graph of velocity versus time. We'll refer to this graph as graph A.



b) Sketch a graph of speed versus time. We'll refer to this graph as graph B.



c) If distance = d = f(t), then

i) Graph A is the graph of

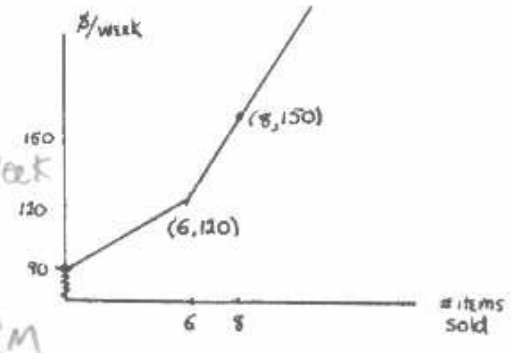
a) $y = f'(t)$ b) $y = f'(|t|)$ c) $y = |f'(t)|$

ii) Graph B is the graph of

a) $y = f'(t)$ b) $y = f'(|t|)$ c) $y = |f'(t)|$

6. (8 points) The graph below indicates the salary scheme at Company A for a certain job.

a) In words, tell a story about the job associated with the graph below. What is the significance of the bend in the graph? Interpret the S intercept and both of the slopes.



The starting salary is \$90/week. For each item sold, there is a salary increase of $\frac{120-90}{6} = \$5/\text{week}$. After 6 items have been sold, the salary increase per item sold jumps to $\frac{150-120}{8-6} = \$15/\text{week}$.

b) The pay scheme at company B for the analogous job is as follows: workers get paid \$80 per week plus and additional \$10 for each item sold. How many items must a worker sell over the course of a week in order to have the job at Company B to be to her advantage? (Since this question is given in words, please use words in giving your answer.)

The worker must sell greater than 2 items and fewer than 8 items per week.

$(90+5x = 80+10x \Rightarrow x=2)$

7. (10 points) Using the limit definition of derivative, find f '(3) if $f(x) = \frac{x}{x-5}$.

The answer is -5/4. We want you to state the limit definition and work through it slowly in a step by step manner.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{x+h}{x+h-5} - \frac{x}{x-5}}{h}$$

$$= \lim_{h\to 0} \frac{x^2+hx-5x-5h-x^2-hx+5x}{(x^2+hx-5x-5x-5h+25)h} = \frac{-5}{x^2-10x+25}$$

$$= \frac{-5}{4}$$

at $x=3$

8. (9 points) Let $f(x) = \begin{cases} (x-3)^2 & \text{if } x < 3 \\ 3x + b & \text{if } x \geq 3 \end{cases}$ where b is the number such that the limit as x -> 3 of f(x) exists.

a) Find b.

let $f_1(x) = (x-3)^2$, $f_2(x) = 3x + b$

$f_1(3) = 0 = f_2(3) = 9 + b$ so $b = -9$

b) i) What is f '(6)?

$f'(6) = f_2'(6) = 3$

ii) What is f '(0)?

$f'(0) = f_1'(0) = 2x - 6 = -6$

c) What is f ' (3)? Please explain your answer clearly and precisely.

$f'(3)$ DNE. $f_1'(3) = 0$, $f_2'(3) = 3$, So the limit $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ when $x = 3$ is different if h approaches 0 from the right or from the left. So the limit, and thus the derivative, does not exist. (at x=3, we're at a corner)

9. (10 points) A size of a herd of hyenas grows at a rate of 20% every 4 years. Let H_0 denote the number of hyenas today.

a) If t denotes the number of years from today, write a function H(t) that tells us the size of the herd t years from today.

$H(t) = H_0 \cdot (1.2)^{t/4}$

b) Which of the following statements is true? How do you know?

i) The herd grows at a rate of 5% per year.

ii) The herd grows at a rate of more than 5% per year.

iii) The herd grows at a rate of less than 5% per year.

compare: $H_0 \cdot (1.2)^{t/4}$ and $H_0 \cdot (1.05)^t$ when $t=1$: $1.2^{1/4} = 1.0466$, $1.05^1 = 1.05$

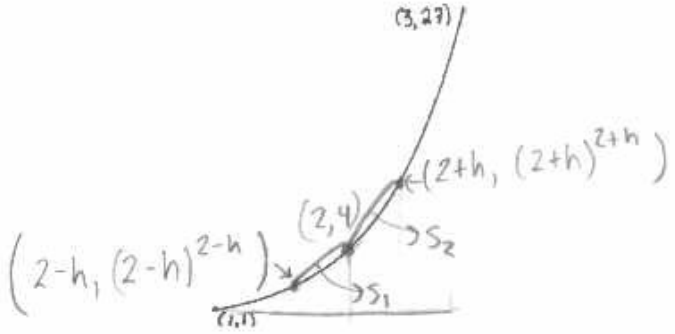
so [iii] is true

c) By what percent does the herd size increase every month?

$(1.2)^{\frac{1}{12 \cdot 4}} = 1.0038$ so it grows by .38% each month.

10. (10 points) $f(x) = x^x$. The graph of $y = f(x)$ is given.

a) Label the x and y coordinates of the points on the graph with x coordinates 2, 2 + h, and 2 - h.



b) Approximate f '(2) by using secant lines. Please give both an upper and a lower bound for f '(2). These upper and lower bounds should differ by no more than 1/10.

let $h = .01$

$f'(2) \approx \dfrac{2^2 - (1.99)^{1.99}}{.01} = 6.706 =$ slope of S_1 (lower bound)

$f'(2) \approx \dfrac{2.01^{2.01} - 1.99^{1.99}}{.02} = 6.773 =$ slope of secant line connecting $(2.01, 2.01^{2.01})$ and $(1.99, 1.99^{1.99})$

This is an upper bound for the slope b/c the function is convex

Upper bound for f ' (2) 6.773

Lower bound for f ' (2) 6.706

11. (10 points) I work an h hour day. 1/x of my time is spent on the road. The remainder is spent in consultation. I recieve A $/hour as a consultant. I receive no money when I'm on the road. In fact, each day I pay $ G in gas and tolls and I estimate that each day costs C cents in wear and tear on the car. I have no other expenses. What is my daily profit?

Note: if x = 5, I spend 1/5 of my work day on the road and the rest of my work day is spent in consultation.

In order to get full credit for your answer you must have **words** on this page. Your work must be well-organized and laid out.

profit = revenue - cost

$= \left(h - \frac{1}{x}\right) \cdot A - G - \frac{C}{100}$

time spent consulting x $/hour

gas $

wear on the car