Solutions to The Second Math Xa Exam

December 9, 1996

#1 a) $f(x) = -\frac{x^5}{\sqrt{5}} + \pi e^x + e^\pi + 3(5^x)$

$= -\frac{1}{\sqrt{5}} x^5 + \pi e^x + e^\pi + 3(5^x)$

$\boxed{f'(x) = -\frac{5}{\sqrt{5}} x^4 + \pi e^x + 3\ln 5 \cdot 5^x}$

Common Error: e^π is a constant. It's deriv. is 0.

b) $g(x) = \ln(4\sqrt{\pi}\, x^3 e^{x/3})$

$= \ln 4\sqrt{\pi} + 3\ln x + \ln e^{x/3}$

$= \underbrace{\ln 4\sqrt{\pi}}_{\text{constant}} + 3\ln x + x/3$

$\boxed{g'(x) = \frac{3}{x} + \frac{1}{3}}$

c) $h(x) = \underbrace{3x}_{f}\,\underbrace{e^{-4x}}_{g}$ Use product rule

$h'(x) = f'g + g'f$

$= 3e^{-4x} + (\ln e^{-4}) e^{-4x} \cdot 3x$

$\boxed{h'(x) = 3e^{-4x} - 12x e^{-4x}}$

Common Error: $(fg)' \neq f'g'$

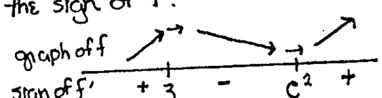
#2. a) Critical pts of f: $\begin{cases} f' = 0 \\ f' \text{ undef.} \\ \text{endpts. of domain of } f \end{cases}$

Domain has no endpts & f' is undef. on the domain of f, so we look at $f' = 0$

$(x-3)(\ln x - 2) = 0$

$x = 3$ or $\ln x = 2$, $x = e^2$ $\boxed{\text{critical pts: } x=3,\ x=e^2}$

b) Approach #1: Use a # line and look at the sign of f'!



From this you can tell that there's a local maximum at $x = 3$ & a local minimum at $x = e^2$.

Approach #2: Look at the sign of $f''(x)$

$f'(x) = (x-3)(\ln x - 2)$ use the product rule

(if you multiply out, you still must use the product rule on $x \ln x$

$f''(x) = (x-3)(\frac{1}{x}) + 1(\ln x - 2)$

$= 1 - \frac{3}{x} + \ln x - 2$

$= -\frac{3}{x} + \ln x - 1$

$x=3$: $f''(3) = -\frac{3}{3} + \ln 3 - 1 = -2 + \ln 3 < 0$

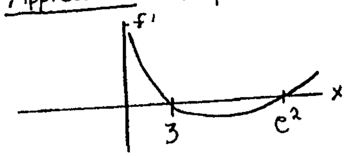
$f'' < 0 \Rightarrow f$ is concave down $\Rightarrow$ local max at $x=3$

$x=e^2$: $f''(e^2) = -\frac{3}{e^2} + \ln e^2 - 1 = -\frac{3}{e^2} + 2\ln e - 1$

$= -\frac{3}{e^2} + 1 > 0$

$f'' > 0 \Rightarrow f$ is concave up $\Rightarrow$ local min at $x = e^2$.

Approach #3: Graph f': We find



f' changes from (+) → (-) at x=3 so f changes from increasing to decreasing local max at x=3

f' changes from (-) → (+) at $x = e^2$ so f changes from decreasing to increasing local min at $x = e^2$

c) $m = f'(1) = (1-3)(\ln 1 - 2)$
$= (-2)(0-2) = 4$

tangent line: $y = 4x + b$

(1,4) lies on the line

$4 = 4(1) + b$

$0 = b$

$\boxed{y = 4x}$

Error: the slope of the line is a constant! It is $f'(1)$ NOT $f'(x)$.

#3. $f(x) = \frac{1}{3}x^3 + x + 1$

a) $f'(x) = x^2 + 1$

$f'(x) > 0$ for all x

$\Rightarrow$ $\boxed{\text{f always increasing}}$

b.) $f''(x) = 2x$

	concave down		concave up
f''	−	0	+

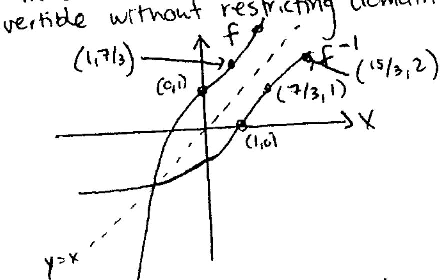
f changes concavity at x=0

$\Rightarrow (0, f(0)) = \boxed{(0,1) \text{ inflection point}}$

Common Error: Just because $f''(0) = 0$ does not necessarily make x=0 an infl. point.

c.) Since f is always increasing it must be 1-1. Thus, f is invertible without restricting domain.

d.)



e.) $f^{-1}(f(x)) = x$ for any $x \in$ domain of f

$\Rightarrow$ $\boxed{f^{-1}(f(0)) = 0}$

f.) f(x) is always increasing, so f(x) = k has $\boxed{\text{one solution for any k.}}$

g.) f(x) always increasing

$\Rightarrow$ max value = $f(1) = \boxed{7/3}$

min value = $f(-2) = \boxed{-11/3}$

#4. $V = \pi r^2 h \quad 108 = 2\pi r + h$

Get V in terms of one variable:

$V = \pi r^2(108 - 2\pi r)$

$= 108\pi r^2 - 2\pi^2 r^3$

$\frac{dV}{dr} = 216\pi r - 6\pi^2 r^2$

$= 6\pi r(36 - \pi r)$

$\frac{dV}{dr} = 0 \Rightarrow r = 0, \; r = \frac{36}{\pi}$

f: ↗ ↘

f': (endpt.) 0 + $\frac{36}{\pi}$ −

So, $r = \frac{36}{\pi}$ is local min.

$h = 108 - 2\pi r = 108 - 2\pi\left(\frac{36}{\pi}\right) = 36$

$\boxed{\text{radius} = \frac{36}{\pi}, \; \text{height} = 36}$

#5.

a) At \$250 per seat, 300 people will buy tickets.

b) At \$275 per seat, each \$1 increase in price means 1 fewer passenger.

c) \$220 is the price at which 275 people will buy tickets.

d) Raising the current price by \$50 will result in 190 tickets sold.

e) If the current price is tripled, only half as many tickets will be purchased.

Common Error: It confuses cause and effect to write "If the number of tickets sold goes down by 50%, the price will triple." It is the change in price that causes the change in sales.

#6.

a) $1 = 3\ln(2x+e)$

$\frac{1}{3} = \ln(2x+e)$

$e^{1/3} = 2x + e$

$e^{1/3} - e = 2x$

$\boxed{x = \frac{e^{1/3} - e}{2}}$

Common Error: $\ln(2x+e) \neq \ln 2x + \ln e$! We have no log rule for simplifying $\log(A+B)$.

b) $\log x^2 = (\log x)^2$

$2\log x = (\log x)^2$

$0 = (\log x)(\log x) - 2\log x$

$0 = (\log x)(\log x - 2)$

$\Rightarrow \log x = 0$ or $\log x = 2$

$\boxed{x = 1 \text{ or } x = 100}$

Common Error: If you divide by $\log x$, you are assuming $\log x \neq 0$ to avoid division by zero. Thus, you must check to see if $\log x = 0$ is a soln. It is.

#7 a) $P(t) = 240(2^{t/10})$

b) $400 = 240(2)^{t/10}$

$\frac{40}{24} = 2^{t/10}$

$5/3 = 2^{t/10}$

$\ln(5/3) = t/10 \ln 2$

$t/10 = \frac{\ln(5/3)}{\ln 2}$

$t = \frac{10\ln(5/3)}{\ln 2}$

$t \approx 7.37$ years.

c) $P'(t) = 240\ln(2^{1/10}) \cdot 2^{t/10}$

$P'(t) = 24\ln 2 \cdot 2^{t/10}$

We want to solve

$12 = 24\ln 2 \cdot 2^{t/10}$

since the rate of change of fish ($\frac{dP}{dt}$ or $P'(t)$) is supposed to ℓ ?.

$12 = 24\ln 2 \cdot 2^{t/10}$

$\frac{1}{2\ln 2} = 2^{t/10}$

$\ln\left(\frac{1}{2\ln 2}\right) = t/10 \ln 2$

$\frac{10\ln\left(\frac{1}{2\ln 2}\right)}{\ln 2} = t$

(or $t = \frac{-10\ln(2\ln 2)}{\ln 2}$

$= \frac{-10\ln(\ln 4)}{\ln 2}$)

There are many variations on this answer.

$t \approx -4.7$ yrs.

This means the fish population was growing at a rate of 12 fish/yr. about 4.7 yrs. ago.

This makes sense: right now fish are growing at a rate ... ≈ 16.3 fish/...

d) $240 \cdot 2^{t/10} = 240e^{kt} \Rightarrow 2^{1/10} = e^k$

$k = \ln 2^{1/10} = \frac{1}{10}\ln 2 \approx .069$

$P(t) = 240e^{.069t}$

#8. a) $f(x-2)$: $\boxed{\#6}$ shift rt. 2 units

b) $-f(x)+2$: $\boxed{\#4}$ flip over x-axis, move ↑ 2

c) $f(2x)$: $\boxed{\#5}$ squished horizontally by a factor of 2

d) $f(\frac{x}{2})$: $\boxed{\#7}$ stretched horizontally by a factor of 2.

Common Errors $f(2x)$: replacing x by $2x$ squishes, & does NOT stretch. x can be ½ as big as it used to be to get the same y-value.

Try values!! E.g. for a) try $x=2$: $f(2-2) = f(0) = 0$ so a) can't correspond to #1!

#9. a) $\boxed{y = \frac{2x^2(x-4)}{(x-3)^2(x+2)}}$ b) $\boxed{y = \frac{-(x^2)(x-4)}{(x+2)(x-3)}}$

c) $y = Kb^x + 2$ basic form

(0,0) is on the graph so

$0 = Kb^0 + 2$

$0 = K + 2 \Rightarrow K = -2$

$y = -2b^x + 2$

(1,1) is on the graph, so

$1 = -2b + 2$

$-1 = -2b$

$b = ½$

$\boxed{y = -2(\frac{1}{2})^x + 2}$

#10. i) a) True : $P(x)$ is cts. and looks like ↗ or ↖ so it must cut the x-axis at least once.

b) Not necessarily. Could have 5 roots

c) Not necessarily could be x^5: No turning pts.

d) True : P' is a 4^{th} degree polynomial. The turning pts. of P are pts at which the graph of P' cuts through the x-axis — could be 4 times but no more.

ii) a) Not an absolute min since ... $\pm\infty$ either as $x \to \infty$