Due 12/1/04

11.1

Problem 3.
Answers will vary; one possibility is: $f(x) = -(x-1)^3$.

Problem 4.
Answers will vary; two possibilities are: $f(x) = \frac{1}{3}x^3 - x^2$ and $f(x) = (x+1)(x-2)^2$.

Problem 5.
Answers will vary; one possibility is: $f(x) = (x-1)^3 = x^3 - 3x^2 + 3x - 1$.

Problem 6.
Answers will vary; one possibility is: $f(x) = x^3 + x$.

Problem 7.
Answers will vary; one possibility is: $f(x) = -\frac{2}{27}(x-3)^3$. As f is always decreasing, f must have exactly one zero. Hence, f is of the form $f(x) = k(x-3)^3$, where $k < 0$ is a constant. Using $f(0) = 2$, we have that $2 = k(0-3)^3$, from which we obtain $k = -\frac{2}{27}$.

Problem 12.

(a) $f(x) = x^3 - 3x + 1 \Rightarrow f'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x+1)(x-1)$. $0 = f'(x) \Rightarrow x = -1$ and $x = 1$ are the critical points. Now $f''(x) = 6x$ gives $f''(-1) = -6 < 0$ and $f''(1) = 6 > 0$. Therefore, $x = -1$ is a local maximum point and $x = 1$ is a local minimum point. Moreover neither of these critical points are absolute extreme points because $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

(b) $f(x) = x^3 + 3x + 1 \Rightarrow f'(x) = 3x^2 + 3 = 3(x^2+1)$. $0 = f'(x) \Rightarrow x^2 = -1$. Hence there are no real values of x for which $0 = f'(x)$ and consequently no critical points.

Problem 13.

(a) $f(x) = -x^3 - 3x^2 + 9x + 5 \Rightarrow f'(x) = -3x^2 - 6x + 9 = -3(x+3)(x-1)$. Now $0 = f'(x) \Rightarrow x = -3$ or $x = 1$. Now $f''(x) = -6x - 6$, and hence $f''(-3) = 12 > 0$ and $f''(1) = -12 < 0$. Thus there is a local minimum at $x = -3$ and a local maximum at $x = 1$. As $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$, neither of the local extrema are absolute extrema.

(b) $f(x) = x^3 + 3x^2 + 9x + 8 \Rightarrow f'(x) = 3x^2 + 6x + 9 = 3(x^2 + 2x + 3)$. Now $0 = f'(x) \Rightarrow x = \frac{-2\pm\sqrt{2^2-4(1)(3)}}{2}$. Hence there are no real values of x for which $0 = f'(x)$ and consequently no critical points.

11.2

Problem 2.
Answers will vary; one possibility: $P(x) = (x-1)(x+3)$.

Problem 3.
Answers will vary; one possibility: $P(x) = x(x+1)(x-5)$.

Problem 4.
Answers will vary; one possibility: $P(x) = x^4 + 5$.

Problem 5.
Answers will vary; one possibility: $P(x) = (x - \sqrt{2})^4$.

Problem 6.
Answers will vary; one possibility: $P(x) = x(x-9)^2(x-3)(x+e)$.

Problem 7.
Answers will vary; one possibility: $P(x) = -\frac{(x-\pi-1)^3}{(\pi+1)^3}$.

Problem 8.
Answers will vary; one possibility: $P(x) = (x+3)^4 + 2$.

Problem 9.
$P(x) = k(x-2)^2(x+3)^2$, for some $k \neq 0$. Now $-2 = P(0) = k(4)(9) = 36k \Rightarrow k = -\frac{1}{18}$. This answer is unique.

Problem 15.

(a) $f(x) = 2x^3 + 2x^2 - 12x = 2x(x+3)(x-2)$. The zeros are $x = -3, 0$, and 2.

(b) $g(x) = 2x^3 + 2x^2 + 12x = 2x(x^2 + x + 6)$. Now if $x^2 + x + 6 = 0$, then $x = \frac{-1 \pm \sqrt{1-4(6)}}{2} = \frac{-1 \pm \sqrt{-23}}{2}$, which are not real solutions. The only zero is $x = 0$.

Problem 17.

(a) $P(x) = x^3 - x^2 - 4x + 4 = (x^2 - 4)(x - 1) = (x + 2)(x - 2)(x - 1)$. The zeros are $x = -2, 1$, and 2.

(b) $Q(x) = x^3 - x^2 + 4x - 4 = (x - 1)(x^2 + 4)$. Now $x^2 + 4 = 0$ has no real solutions; hence, the zero is $x = 1$.

Problem 22.

(a) Definitely false. A polynomial that is symmetric about the origin has odd degree.

(b) Definitely true. A polynomial that is symmetric about the origin has odd degree.

(c) Definitely false. A polynomial that is symmetric about the origin has odd degree, and odd-degree polynomials have at least one zero.

(d) Definitely true. A polynomial that is symmetric about the origin has odd degree, and odd-degree polynomials have at least one zero.

(e) Definitely true. Because $P(x)$ is symmetric about the origin, $(0, 0)$ cannot be a turning point, and if $(a, P(a))$ is a turning point, $(-a, -P(a))$ is also a turning point. Hence the number of turning points is a (nonegative) even number.

Problem 23.

(a) Definitely false. An even-degree polynomial can be symmetric about the y−axis but not the origin.

(b) Possibly true. True if $P(x) = x^4$; false if $f(x) = x^4 + x^3$.

(c) Definitely true. $\lim_{x\to-\infty} P(x) = \lim_{x\to\infty} P(x) = L$, where $L = \pm\infty$. As $P(0) = 0$, $P(x)$ must have at least one turning point.

(d) Definitely false. $\lim_{x\to-\infty} P(x) = \lim_{x\to\infty} P(x) = L$, where $L = \pm\infty$, which implies that $P(x)$ must have an odd number of turning points.

(e) Definitely true. $\lim_{x\to-\infty} P(x) = \lim_{x\to\infty} P(x) = L$, where $L = \pm\infty$, which implies that $P(x)$ must have an odd number of turning points.

11.3

Problem 2.

(a) Since $P'(x)$ is degree 5, P is degree 6.

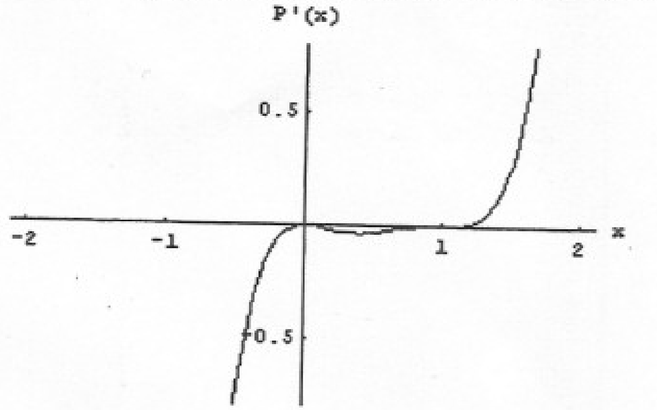
(b) The critical points of P occur whenever $P'(x) = 0$, that is at $x = 0, -2$.

(c) Because $\lim_{x\to-\infty} P'(x) = -\infty$, $\lim_{x\to\infty} P'(x) = \infty$, and $P'(x)$ is continuous everywhere, $P(x)$ has an absolute minimum. We know it must be obtained at one of the critical points, so we calculate $P''(x) = 2x(x+2)^3 + 3x^2(x+2)^2$; unfortunately, $P''(0) = 0$ and $P''(-2) = 0$, so the second derivative test lends no information. (How could you have predicted that?) Instead we look at the sign of $P'(x)$; across $x = 0$ the sign remains positive, but across $x = -2$ the sign changes from negative to positive. Thus, there is an absolute minimum at $x = -2$. However, we cannot find the value of the function here since we don't have an expression for the function; we can determine only the value of the function relative to some other value of the function.

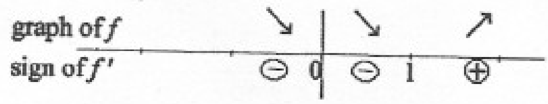
pg3

Problem 4.

(a) Note that $P'(x)$ has two zeros, at $x = 0, 1$; these are the critical points of $P(x)$.

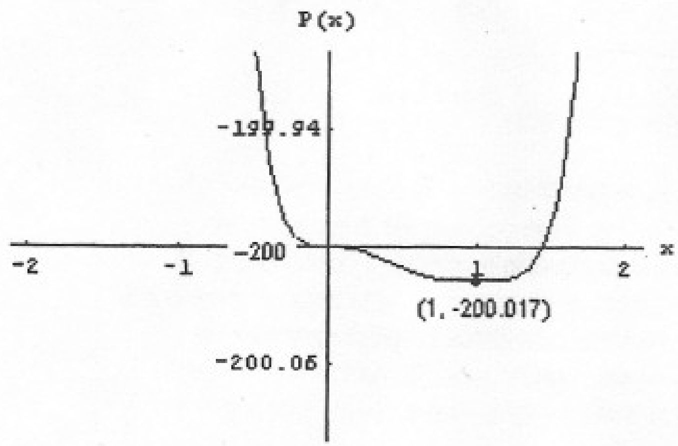


(b)



Note that $P' > 0$ on $(1, \infty)$ and $P' < 0$ on $(-\infty, 0) \cup (0, 1)$.

(c) Because $P(x) < 0$ at $x = 0$, $P(x) < 0$ at $x = 1$ since P is decreasing on the interval. Since $P'(x)$ is positive and increasing for $x > 1$, $P(x)$ must be increasing at an increasing rate for $x > 1$; because $P(x) < 0$ at $x = 1$, $P(x)$ must cross the x-axis.



Problem 5.

(a) False; (b) True; (c) Possible; (d) True; (e) False; (f) False; (g) True; (h) False.

Problem 9.

(a) The graph has the general shape of the graph of a fourth-degree polynomial function. There is a even order zero at $x = a$ and odd order zeros at $x = b$ and $x = c$. Hence the function has the form $P(x) = k(x-a)^2(x-b)(x-c)$, for some constant k. As the $y-$intercept is d, we have $d = P(0) = k(0-a)^2(0-b)(0-c) = ka^2bc \Rightarrow k = \frac{d}{a^2bc}$. Therefore, $P(x) = \frac{d}{a^2bc}(x-a)^2(x-b)(x-c)$

(b) The graph has the general shape of the graph of a sixth-degree polynomial function. There are a even order zeros at $x = a$ and $x = c$ and odd order zeros at $x = b$ and $x = 0$. Hence the function has the form $P(x) = kx(x-a)^2(x-b)(x-c)^2$, for some constant k. As the graph contains the point $(-1, 3)$, we have $3 = P(-1) = k(-1)(-1-a)^2(-1-b)(-1-c)^2 \Rightarrow k = \frac{3}{(a+1)^2(b+1)(c+1)^2}$. Therefore, $P(x) = \frac{3}{(a+1)^2(b+1)(c+1)^2}x(x-a)^2(x-b)(x-c)^2$.

11.4

Problem 1

(a) The graph has a single simple zero at $x = 0$ and a vertical asymptote at $x = -1$. As the sign of y changes across the vertical asymptote, there is an odd power of $(x+1)$ in the denominator of the function. As there is a horizontal asymptote at $y = 2$, the degrees of the numerator and denominator of the function are equal, and the lead coefficient of the numerator is 2. Hence $y = \frac{2x}{x+1} = 2 - \frac{2}{x+1}$.

(b) The graph has simple zeros at $x = -2$ and $x = 0$ and a vertical asymptote at $x = -1$. As the sign of y does not change across the vertical asymptote, there is an even power of $(x+1)$ in the denominator of the function. As there is a horizontal asymptote at $y = 2$, the degrees of the numerator and denominator of the function are equal, and the lead the coefficient of the numerator is 2. Hence $y = \frac{2x(x+2)}{(x+1)^2} = -\frac{2}{(x+1)^2} + 2$.