Name: SOLUTIONS

Math Xa Midterm Examination II—Thursday, December 9, 2004

Please circle your section:

Thomas Judson	Angela Vierling-Claassen	David Harvey	Ben Weinkove
Jennie Schiffman (CA)	Connie Zong (CA)	Margaret Barusch (CA)	Evan Hepler-Smith
MWF 11–12	MWF 10–11	MWF 10–11	MWF 11–12

Problem Number	Possible Points	Score
1	10	
2	5	
3	5	
4	12	
5	5	
6	7	
7	9	
8	12	
9	10	
10	10	
11	15	
Total	100	

Directions—Please Read Carefully! You have two hours to take this midterm. Make sure to use correct mathematical notation. Any answer in decimal form must be accurate to three decimal places, unless otherwise specified. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You do not have to answer the problems in any particular order, so move to another problem if you find you are stuck or spending too much time on a single problem. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit (except if the directions for that question specifically say no justification is necessary, such as in a True/False section). Please be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. You may use a calculator on this exam, but no other aids are allowed. **Good Luck!!!**

1. (10 points) The following are all functions of time. Which are continuous functions, and which are not? If you think a function is not continuous, briefly explain why.

   (a) The height above sea level of a particular airplane flying from Boston to New York, where $t = 0$ is the time that the airplane left Boston.

   Continuous.

   (b) The number of passengers on a particular bus, where $t = 0$ is the time when the driver begins her route.

   Not continuous. The number of passengers on a bus changes in whole number increments.

   (c) The percentage of Harvard professors who are married, where $t = 0$ is January 1, 1636.

   Not continuous. The percentage changes with discrete jumps.

   (d) The distance from the Moon to the Sun, where $t = 0$ is January 1, 1636.

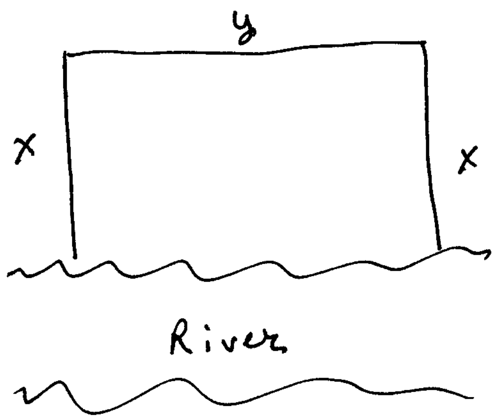
   Continuous.

   (e) The distance from John Harvard's statue to the tallest building in the world, where $t = 0$ is January 1, 1884.

   Not continuous. This distance changes with large jumps.

2. (5 points) Wally, the Welsh farmer, wants to fence off a rectangular area in his field for his sheep to graze in. He wants the biggest area possible. His field is very large, but he only has 200 ft of fencing. However, a straight river runs through his field and his sheep are afraid of water, so he can save some fencing by using the river as one of the sides of the rectangle. What is the largest area that Wally can make for his sheep to graze in?

You must justify your answer carefully for full credit.



$A = xy$ and $2x + y = 200$

$\Rightarrow A = x(200 - 2x) = 200x - 2x^2$

$\Rightarrow A' = 200 - 4x = 0$

$\Rightarrow x = 50$ is a critical point

Since $A'' = -4 < 0$, a maximum occurs at $x = 50$

$\Rightarrow A(50) = 5000$ ft^2 is the largest area.

3. (5 points) Find the derivative of

$$f(x) = \frac{1}{2\sqrt{x}}$$

using the definition of the derivative.

Show all work to receive full credit. You must calculate a difference quotient and take a limit. You will receive no credit for simply applying the power rule.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2\sqrt{x+h}} - \frac{1}{2\sqrt{x}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{2\sqrt{x+h}\sqrt{x}\cdot h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{2h\sqrt{x+h}\sqrt{x}} \quad \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h\to 0} \frac{x - (x+h)}{2h[x\sqrt{x+h} + (x+h)\sqrt{x}]}$$

$$= \lim_{h\to 0} \frac{-h}{2h(x\sqrt{x+h} + (x+h)\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{-1}{2(x\sqrt{x+h} + (x+h)\sqrt{x})}$$

$$= \frac{-1}{2(2x\sqrt{x})} = \boxed{-\frac{1}{4x\sqrt{x}}}$$

4. (12 points) It has been observed that computer technology improves exponentially over time. In fact, *Moore's Law* states that computing power doubles every $1\frac{1}{2}$ years. Suppose that you have a typical desktop computer that can carry out 10^9 calculations per second, and assume that Moore's Law applies to your computer.

(a) Write down a function that gives the number of calculations per second that your computer can carry out, t years from now.

$$\boxed{C(t) = 10^9 \cdot 2^{t/1.5}}$$

(b) By what percentage does the speed of your computer increase each week? (Assume that each year has exactly 52 weeks.)

$$C(1/52) = 10^9 \cdot 2^{(1/52)/1.5} \approx 10^9 \cdot 1.0089261$$

$$\Rightarrow \quad 0.89\ \%$$

(c) By studying the behavior of neurones (the cells in human brains), scientists have estimated that the typical human brain carries out the equivalent of 10^{14} calculations per second. Will your computer be calculating more quickly than your brain in 20 years time? What about in 30 years time?

$$C(20) = 10^9 \cdot 2^{20/1.5} \approx 1.032 \times 10^{13} \quad \boxed{\text{No}}$$

$$C(30) = 10^9 \cdot 2^{30/1.5} \approx 1.049 \times 10^{15} \quad \boxed{\text{Yes}}$$

5. (5 points) Estimate $\frac{1}{1.99}$ using linear approximation. *Hint*: If $f(x) = 1/x$, then $f'(x) = -1/x^2$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,\Delta x$$

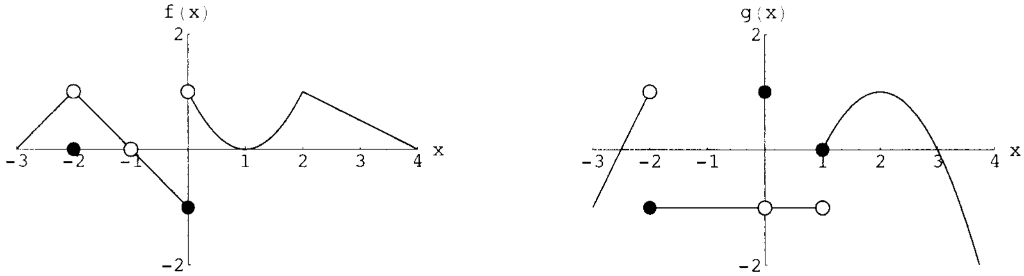
If $x = 2$ and $\Delta x = -0.01$, then

$$\frac{1}{1.99} = f(2 - 0.01) \approx f(2) + f'(2)(-0.01)$$

$$= \frac{1}{2} + \left(-\frac{1}{4}\right)(-0.01)$$

$$= \boxed{0.5025}$$

6. (7 points) Let f and g be the functions whose graphs are given below. Use the graphs to evaluate the following expressions. If the expression cannot be evaluated, say why.



(a) $g(0)$

$g(0) = 1$

(b) $\lim_{x \to 0} g(x)$

$\lim_{x \to 0} g(x) = -1$

(c) $\lim_{x \to 1} [f(x) + g(x)]$

DOES NOT EXIST

(d) $\lim_{x \to 1} [f(x)g(x)]$

$\lim_{x \to 1} f(x)g(x) = 0$

(e) $\lim_{x \to -2^-} f(x)g(x)$

$\lim_{x \to -2^-} f(x)g(x) = 1$

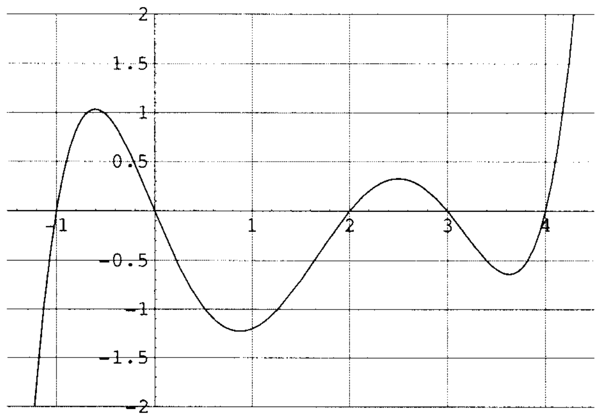
(f) $\lim_{x \to -2^+} f(x)g(x)$

$\lim_{x \to -2^+} f(x)g(x) = -1$

(g) $\lim_{x \to -2} f(x)g(x)$

DOES. NOT EXIST

7. (9 points) The *derivative* $f'(x)$ of a function is graphed below.



(a) Approximate the instantaneous rate of change of the *original function* at $x = 0$.

$f'(0) = 0 \Rightarrow \boxed{x=0}$

(b) At what x-values does the *original function* $f(x)$ have a local minimum?

$x = -1, 2, 4$

the derivative changes from negative to positive.

(c) On what intervals is the *original function* $f(x)$ concave up?

$(-1.2, -0.6)$, $(0.9, 2.5)$, $(3.7, 4.2)$

8. (12 points) After being sent to her room for a "time out," little Suzy opens her bedroom window and throws her favorite toy straight up in the air. She was hoping that the toy would fly, but instead it went up for a little while and then dropped to the ground. The height, in feet, of the toy as a function of time is given by the following equation

$$h(t) = -16t^2 + 2t + 17,$$

where t is measured in seconds and $t = 0$ corresponds to the time that Suzy threw the toy out of the window.

(a) With what velocity did Suzy throw the toy?

$$h'(t) = -32t + 2$$
$$h'(0) = \boxed{2 \text{ ft/sec}}$$

(b) How high off the ground did the toy rise before it started to fall?

$h'(t) = -32t + 2 \qquad 0 = -32t + 2$

$\Rightarrow \quad t = \frac{1}{16}$ at the highest point

$\Rightarrow \quad h\left(\frac{1}{16}\right) = \boxed{17.0625 \text{ ft}}$

(c) How fast was the toy going at the exact moment when it hit the ground?

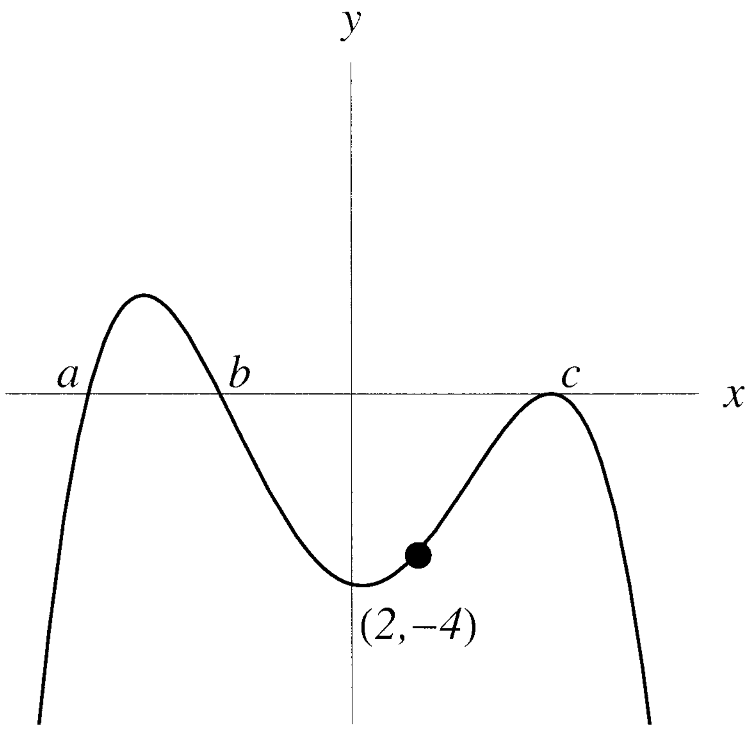
$$h(t) = -16t^2 + 2t + 17 = 0$$
$$\Rightarrow \quad t = \frac{-2 \pm \sqrt{4 - 4(-16)(17)}}{-32} \approx 1.095 \text{ or } -0.970$$
$$h'(1.095) \approx \boxed{-33.05 \text{ ft/sec}}$$

(d) What was the acceleration of the toy at its maximum height?

$$h''(t) = \boxed{-32 \text{ ft/sec}^2}$$

9. (10 points)

(a) Find a polynomial function $P(x)$ whose graph could match the graph below.



$$P(x) = K(x-a)(x-b)(x-c)^2$$

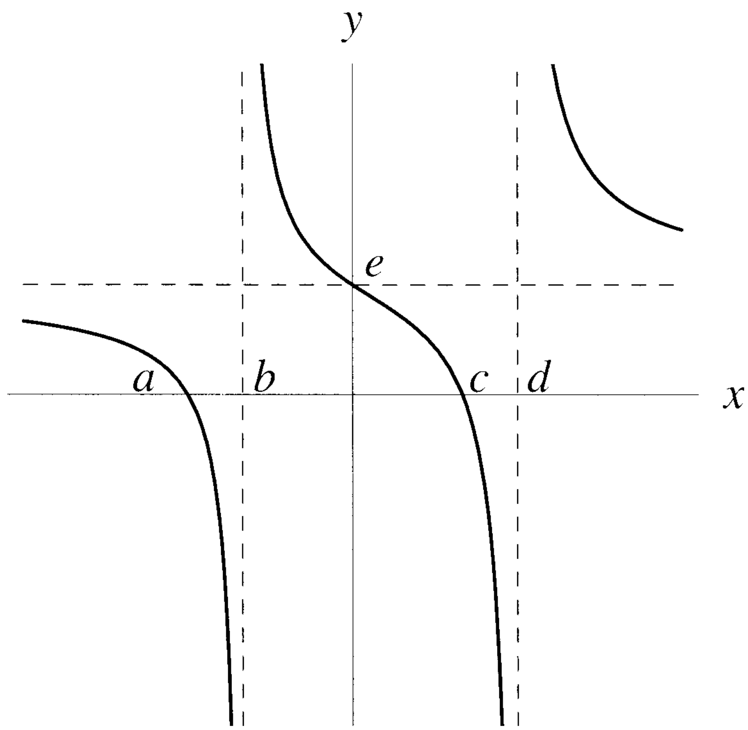
Since $-4 = P(2) = K(2-a)(2-b)(2-c)^2,$

$$K = \frac{-4}{(2-a)(2-b)(2-c)^2}$$

$\Rightarrow$

$$P(x) = \frac{-4(x-a)(x-b)(x-c)^2}{(2-a)(2-b)(2-c)^2}$$

(b) Find a rational function $R(x)$ whose graph could match the graph below.



$$R(x) = \frac{e\,(x-a)\,(x-c)}{(x-b)\,(x-d)}$$

10. (10 points) Suppose that $f(2) = 2$, $f'(2) = -5$, $g(2) = -2$, and $g'(2) = 4$. Find each of the following values.

(a) $h'(2)$, if $h(x) = 2f(x) - 3g(x)$

$$h'(x) = 2f'(x) - 3g'(x)$$
$$h'(2) = 2f'(2) - 3g'(2) =$$
$$= 2(-5) - 3(4) = \boxed{-22}$$

(b) $h'(2)$, if $h(x) = \sqrt{x}f(x) - e^x g(x)$

$$h'(x) = \sqrt{x}\,f'(x) + \frac{1}{2\sqrt{x}}f(x) - e^x g(x) - e^x g'(x)$$
$$h'(2) = \sqrt{2}\,f'(2) + \frac{1}{2\sqrt{2}}f(2) - e^2 g(2) - e^2 g'(2)$$
$$= \sqrt{2}(-5) + \frac{1}{2\sqrt{2}}(2) - e^2(-2) - e^2(4)$$
$$= -5\sqrt{2} + \frac{1}{\sqrt{2}} + 2e^2 - 4e^2 = \boxed{-5\sqrt{2} + \frac{1}{\sqrt{2}} - 2e^2}$$
$$\approx \boxed{-21.142}$$

(c) $h'(2)$, if $h(x) = f(x)/g(x)$

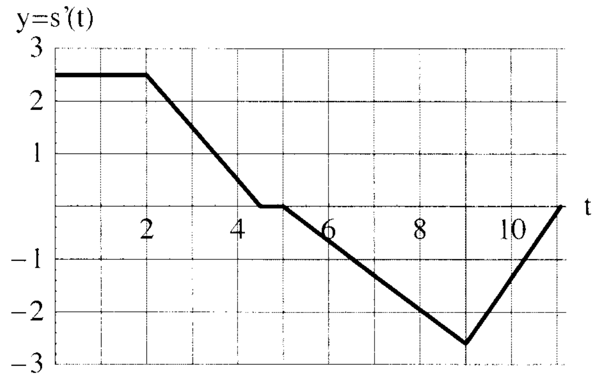
$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$h'(2) = \frac{f'(2)g(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(-5)(-2) - 2(4)}{4}$$
$$= \boxed{\frac{1}{2}}$$

(d) Find an equation of a line tangent to f at $x = 2$.

$$y = f'(2)(x-2) + f(2)$$
$$= -5(x-2) + 2$$
$$\boxed{y = -5x + 12}$$

11. (15 points) You are driving north on a street to a friends house. You realize that you have passed the house, and so you slow down. There are no other cars on the road, so you put the car in reverse and back up, coming to a stop in front of your friend's house. Suppose $s(t)$ gives the position of the car at time t, in minutes, where a positive value for s represents a position to the north of your friend's house. The figure below shows the graph of the *derivative* $y = s'(t)$.



(a) At what time t did you realize you passed the friend's house and start slowing down?

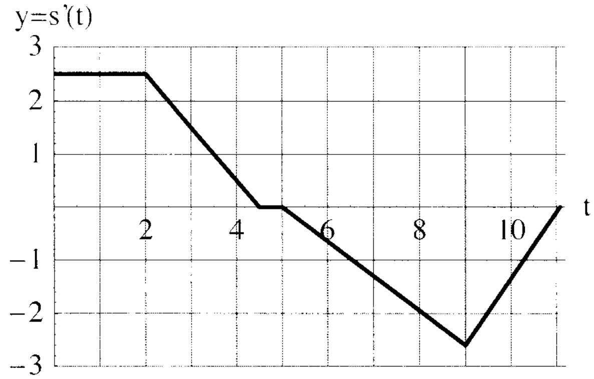
$t = 2$

(b) At what time did you put the car into reverse and start backing up?

$t \approx 5$

(c) At time $t = 3$, were you accelerating, decelerating, or going at a constant speed?

decelerating



(d) Find the approximate value of $s''(10)$. Describe what $s''(10)$ means in words (in terms of what is happening with the car).

$s''(10)$ = slope of $s'(t)$ at $t=10$

$\approx \frac{2.5}{2} = \boxed{5}$

The car is still backing up, but more slowly

(e) Draw a possible graph of the position $s(t)$, given that at time $t = 0$, you were passing your friends house (which is at position $s = 0$).