Name: SOLUTIONS

Math S–Xab Final Exam (Part I)—Tuesday, August 17, 2004

Derek Bruff
Thomas Judson
Evan Hepler-Smith (CA)

Problem Number	Possible Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
Total	40	

Directions—No Calculators are allowed on this part of the exam. Please Read Carefully! You have three hours to take both parts of this exam. Make sure to use correct mathematical notation. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You do not have to answer the problems in any particular order, so move to another problem if you find you are stuck or spending too much time on a single problem. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit (except if the directions for that question specifically say no justification is necessary, such as in a True/False section). Please be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers.

1. (5 points) Find the derivative of $f(x) = \sqrt{2x + \tan x}$

$$f'(x) = \frac{1}{2\sqrt{2x + \tan x}} (2 + \sec^2 x)$$

2. (5 points) Find the derivative of $f(x) = e^x \arctan x$.

$$f'(x) = e^x \arctan x + \frac{e^x}{1 + x^2}$$

3. (5 points) Find the derivative of $f(x) = \dfrac{\sin x}{x^2 - 2x + 5}$.

$$f'(x) = \frac{(x^2 - 2x + 5)\cos x - (2x - 2)\sin x}{(x^2 - 2x + 5)^2}$$

4. (5 points) Find the derivative of $f(x) = \ln\left(\dfrac{x(x^3 - x)^{20}}{\sqrt[7]{x^2 - 2}}\right)$

$$f(x) = \ln x + 20 \ln(x^3 - x) - \frac{1}{7}\ln(x^2 - 2)$$

$$f'(x) = \frac{1}{x} + \frac{20}{x^3 - x}(3x^2 - 1) - \frac{1}{7}\,\frac{2x}{x^2 - 2}$$

5. (5 points) Evaluate the antiderivative $\int \sin^3 x \cos x \, dx$.

Let $u = \sin x$ and $du = \cos x \, dx$

Then

$$\int \sin^3 x \cos x \, dx = \int u^3 du$$

$$= \frac{u^4}{4} + C = \frac{1}{4} \sin^4 x + C$$

6. (5 points) Evaluate the antiderivative $\int \left(x^7 + e^x - \frac{1}{x} + e^\pi \right) dx$.

$$\frac{1}{8} x^8 + e^x - \ln|x| + e^\pi x + C$$

7. (5 points) Evaluate the definite integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$.

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x \Big|_0^1 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

8. (5 points) Evaluate the definite integral $\int_0^{\sqrt{\pi/3}} 2x \sin(x^2)\, dx$.

Let $u = x^2$ and $du = 2x\, dx$.

Then

$$\int_0^{\sqrt{\pi/3}} 2x \sin(x^2)\, dx = \int_0^{\pi/3} \sin u\, du$$

$$= -\cos u \Big|_0^{\pi/3} = -\left(\frac{1}{2}\right) - (-1) = \frac{1}{2}$$

Name: SOLUTIONS

Math S–Xab Final Exam (Part II)—Tuesday, August 17 , 2004

Derek Bruff
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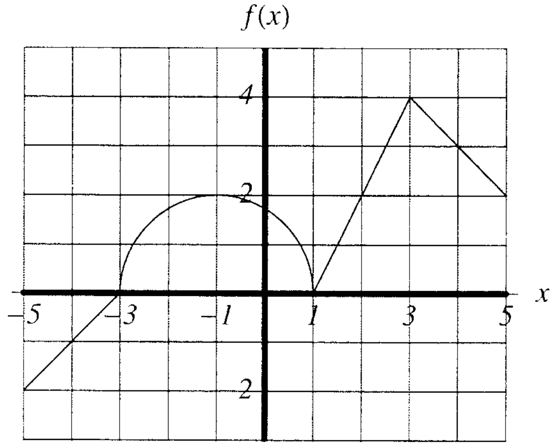
Problem Number	Possible Points	Score
1	20	
2	10	
3	10	
4	10	
5	25	
6	10	
7	20	
Total	105	

Directions—Calculators are allowed on this part of the exam. Please Read Carefully! You have three hours to take both parts of this exam. Make sure to use correct mathematical notation. Any answer in decimal form must be accurate to three decimal places, unless otherwise specified. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You do not have to answer the problems in any particular order, so move to another problem if you find you are stuck or spending too much time on a single problem. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit (except if the directions for that question specifically say no justification is necessary, such as in a True/False section). Please be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers.

1. (20 points) Let $F(x) = \int_{-1}^{x} f(t)\,dt$, where f is the function graphed below. (The graph of f is made up of straight lines and a semicircle.)



(a) Evaluate $F(2)$, $F'(2)$, and $F''(2)$.

$$F(2) = \int_{-1}^{2} f(x)\,dx = \pi + 1$$

$$F'(2) = 2$$

$$F''(2) = 2$$

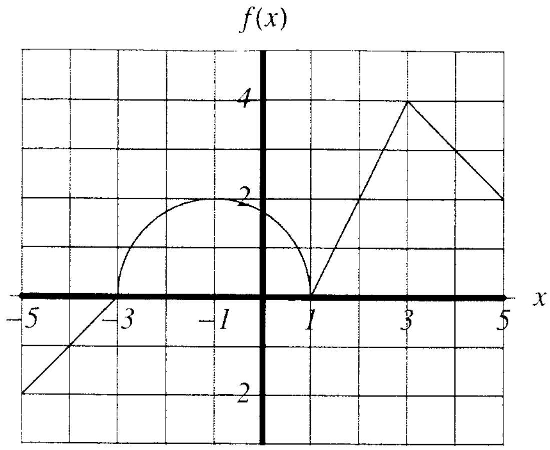
(b) Identify all the critical points of F in the interval $[-5, 5]$.

$x = -3$ and 1

(c) Where in the interval $[-5, 5]$ is F increasing? Justify your answer.

$[-3, 1]$ and $[1, 5]$ since

$$F'(x) = f(x) \geq 0$$



(d) Where in the interval $(-5, 5)$ does F have relative extrema (local maxima and local minima)? Justify your answer.

By the First Derivative Test $x = -3$ is a relative minimum.

(e) Find the an equation of the tangent line to F at $x = 2$.

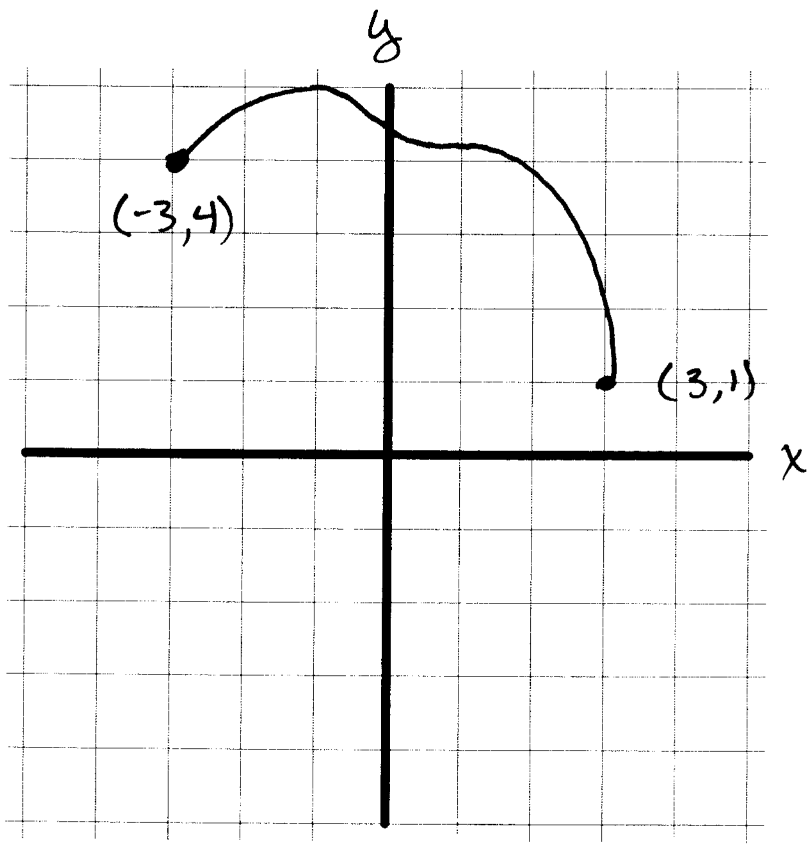
$F(2) = \pi + 1$

$y - (\pi + 1) = 2(x-2) = 2x - 4$

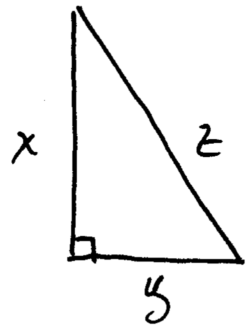
$\boxed{y = 2x - 3 + \pi}$

2. (10 points) Assume that f is a continuous function on the closed interval $[-3, 3]$ with $f(-3) = 4$ and $f(3) = 1$. Also, assume that f' and f'' exist and are continuous on $(-3, 3)$. Use the information in the table below to sketch a possible graph of f.

x	$-3 \le x < -1$	-1	$-1 < x < 0$	0	$0 < x < 1$	1	$1 < x \le 3$
$f'(x)$	$+$	0	$-$	$-$	$-$	0	$-$
$f''(x)$	$-$	$-$	$-$	0	$+$	0	$-$



3. (10 points) A balloon is rising at a constant speed of 5 ft/sec. A boy is cycling along a straight road at a speed of 15 ft/sec. When the boy passes under the balloon, it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 seconds later?



$$x^2 + y^2 = z^2$$

$$\Rightarrow 2xx' + 2yy' = 2zz'$$

$$\Rightarrow xx' + yy' = zz'$$

$x' = 5$

$y' = 15$

$x = 45 + 15 = 60$

$y = 3 \cdot 15 = 45$

$z = \sqrt{60^2 + 45^2} = 75$

$$\Rightarrow 5 \cdot 60 + 15 \cdot 45 = 75 z'$$

$$\Rightarrow z' = \frac{5 \cdot 60 + 15 \cdot 45}{75}$$

$\boxed{13 \text{ ft/sec}}$

4. (10 points) When you cough, your windpipe contracts. The speed v, with which air comes out of your mouth depends on the radius, r of your windpipe. If R is the normal (rest) radius of your windpipe, then for $r \leq R$, the speed is given by

$$v = a(R - r)r^2,$$

where a is a positive constant. What value of r maximizes the speed v? Justify that your value is indeed a maximum.

$$v = a(R - r)r^2 = aRr^2 - ar^3$$

$$\frac{dv}{dr} = 2aRr - 3ar^2 = ar(2R - 3r)$$

$\Rightarrow$ CRITICAL POINTS OCCUR AT

$r = 0$ and $r = \frac{2R}{3}$

We will use the second derivative test to show that $r = \frac{2R}{3}$ is a maximum.

$$\frac{d^2v}{dr^2} = 2aR - 6ar = 2a(R - 3r)$$

$$\left.\frac{d^2v}{dr^2}\right|_{r=\frac{2R}{3}} = 2a\left(R - 3\left(\frac{2R}{3}\right)\right)$$

$$= 2a(R - 2R) = -2aR < 0$$

$\boxed{\text{Thus, } r = \frac{2R}{3} \text{ is a maximum}}$

5. (25 points) The water depth in Moon River x miles downstream from the town of Hardrock is given by

$$D(x) = 20x + 10$$

feet. The width of the river is

$$W(x) = 10(x^2 - 8x + 22)$$

feet. In order to create Moon Lake, a dam is to be built downstream from Hardrock. For engineering reasons, the dam cannot be more than 130 feet high.

(a) For which values of x is $0 \leq D(x) \leq 130$?

$0 \leq x \leq 6$

(b) How far downstream can the dam be built? If the dam were constructed at this point, how wide would it be?

Since $D(6) = 130$, the dam can be at most 6 miles down stream. In this case

$W(6) = 100$ ft

(c) What are the dimensions of the widest dam that could be constructed?

$W'(x) = 10(2x - 8) \Rightarrow \quad x = 4$ is a critical point

$W(0) = 220$

$W(4) = 60$

$W(6) = 100$

$W(0) = 220$

$D(0) = 10$

$W(x)$ achieves its maximum value at $x = 0$

(d) What are the dimensions of the narrowest dam that could be constructed?

$W(x)$ achieves its minimum value at $x = 4$ $\quad (W'(x) = 20(x-4),\ W''(x) = 20)$

$W(4) = 60$

$D(4) = 90$

(e) If the cost of the exam is proportional to the length and the depth of the dam, where should the cheapest dam be located?

$C(x) = k\,W(x)\,D(x) = k(20x + 10)(10(x^2 - 8x + 22))$

$= 100k(2x^3 - 15x^2 + 36x + 22)$

$\Rightarrow C'(x) = 600k(x^2 - 5x + 6) = 600k(x-2)(x-3)$

$\Rightarrow x = 2$ and $x = 3$ are the critical points

$C(0) = 2200k$

$C(2) = 5000k$

$C(3) = 4900k$

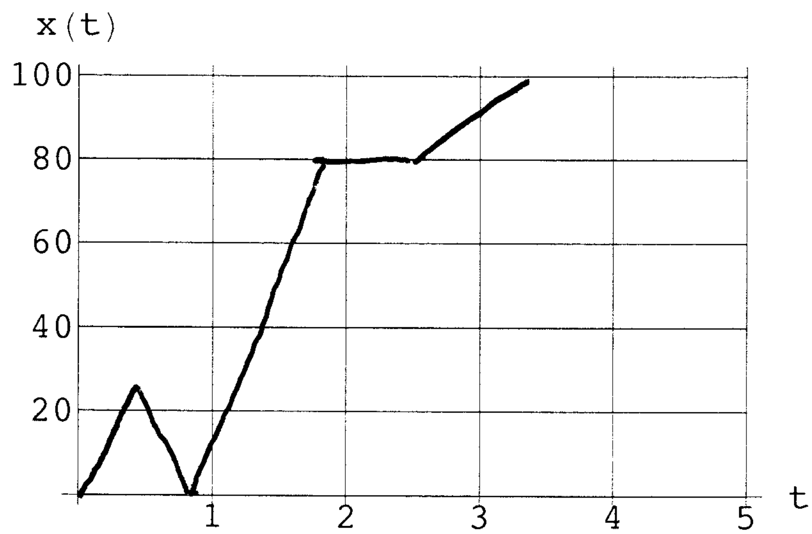
$C(6) = 13000k$

$\Rightarrow C(0)$ is a minimum and the dam should be built $x = 0$ miles downstream.

6. (10 points) The function $x(t)$ describes a trip along a straight road that you made between two cities that are 100 miles apart:

> You left home and drove at 45 mi/hr for a half an hour. You realized that you had left your wallet at home and drove back at 60 mi/hr to get it. Then you drove at 80 mi/hr towards your destination for one hour when a police officer stopped you for speeding. After begging the officer for a half an hour, you were able to get off with just a warning since it was your birthday. You then continued to your destination at 30 mi/hr.

Sketch the graph of the distance x from your starting point (in miles) as a function of time t elapsed (in hours).



7. (20 points) Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right)$$

for $0 \le t \le 30$, where $F(t)$ is measured in cars per minute and t is measured in minutes.

(a) To the nearest whole number, how many cars pass through the intersection over the 30-minute period?

$$\int_0^{30} F(t)\,dt = \int_0^{30} 82 + 4\sin\left(\frac{x}{2}\right)dt$$

$$= 82t - 8\cos\left(\frac{t}{2}\right)\Big|_0^{30} = 2474.08$$

or 2474 cars

(b) Is the traffic flow increasing or decreasing at $t = 7$? Give a reason for your answer.

$F'(7) = -1.872$ or -1.873

Since $F'(7) < 0$, THE FLOW is DECREASING.

(c) What is the average traffic flow over the time interval $10 \leq t \leq 15$? Indicate units of measurement.

$$\frac{1}{5}\int_{10}^{15} F(t)\,dt \approx 81.8992$$

or

82 cars per minute

(d) What is the average rate of change of the traffic flow over the time interval $10 \leq t \leq 15$? Indicate units of measurement.

$$\frac{F(15) - F(10)}{5} \approx 1.51754 \text{ cars}/\text{min}^2$$