Mathematics Xb

Final Examination

May, 24, 1995

After turning in the first part of the exam, you will be able to use your calculator on the second part of the exam. Please note: " because my calculator says so" does not constitute reasoning.

1. (9 points) Find $\frac{dy}{dx}$:

a) $y = x \ln\left(\frac{x}{\sqrt{x}}\right)$ (Simplify your answer.)

$\frac{dy}{dx} =$ _____

b) $y = 2x^{x+1}$

$\frac{dy}{dx} =$ _____

c) $y = \sin^5(x^2) + 3^x$

$\frac{dy}{dx} =$ _____

d) $y = \int_3^x \frac{\sin(t)}{t}\,dt$

$\frac{dy}{dx} =$ _____

2. (8 points) Match the appropriate equation with each graph: (You need not show work.)

A: $y = \dfrac{x^2 - 4}{x^2 - 1}$

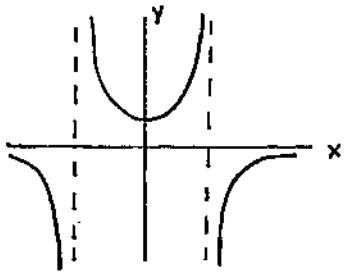
B: $y = \dfrac{-1}{(x - 1)^2(x+1)}$

C: $y = \dfrac{x^4}{x^2 - 1}$

D: $y = \dfrac{x\left(x^2 + 1\right)}{x^2 - 1}$

E: $y = \dfrac{-1}{x^2 - 1}$

(i)



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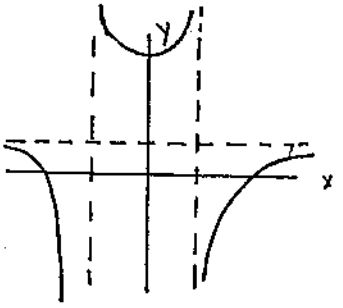

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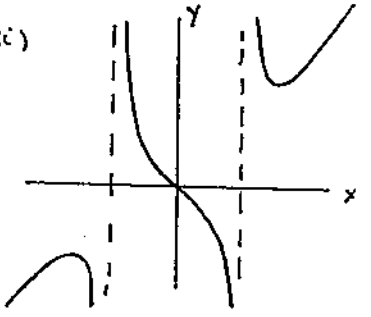

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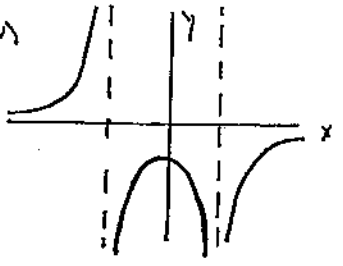

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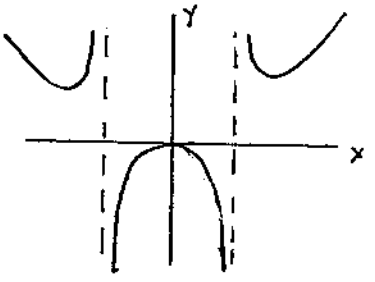


Equation A ; Graph _____
Equation B ; Graph _____
Equation C ; Graph _____
Equation D ; Graph _____
Equation E ; Graph _____

3. (4 points) Put the following definite integrals in ascending order: (Show your work.)

$A = \int_0^{\pi} e^{-x} \sin x \, dx$ $\quad B = \int_0^{2\pi} e^{-x} \sin x \, dx$ $\quad C = \int_{-2\pi}^{-\pi} e^{-x} \sin x \, dx$ $\quad D = \int_{-\pi}^{0} e^{-x} \sin x \, dx$

Answer: _____

4. (12 points) Compute the following integrals:

a) $\int \frac{p}{p+2}\,dp$

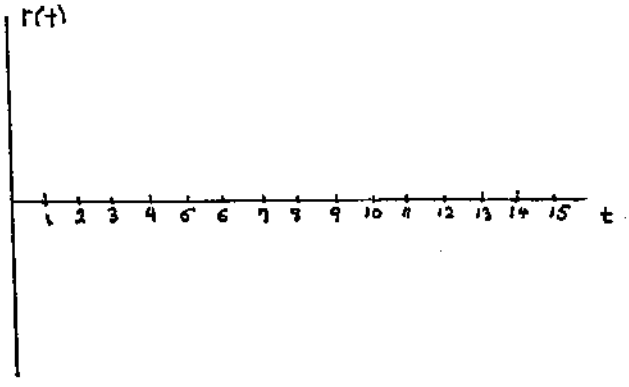
b) $\int_{-3}^{1} |x+2|\,dx$

c) $\int \frac{(\ln x)^2}{x}\,dx$

d) $\int \frac{e^x}{(7+e^x)^3}\,dx$

5. (14 points) The rate of change of water level in a reservoir is given by $r(t) = 1000 \cos\left(\frac{\pi}{6} t\right)$ gallons/month where t is measured in months and t=0 corresponds to January 1, 1994.

a) Draw a graph of r(t) versus t for $0 \le t \le 15$, labelling all r and t intercepts.



b) When in 1994 is the water level in the reservoir the lowest?

c) When (in 1994) is the level of water in the reservoir dropping most rapidly?

d) Compare the water level on July 1st 1994 (t=6) to the water level on January 1 1995. Is it the same? If not, how much higher or lower is it?

e) Compare the water level on October 1st 1994 (t=9) to the water level on July 1st 1995. Is it the same? If not, how much higher or lower is it?

The rate of change of water level in a reservoir is given by $r(t) = 1000 \cos\left(\frac{\pi}{6} t\right)$ gallons/month where t=0 corresponds to January 1, 1994.

f) On January 1st, 1994 there were 3000 gallons of water in the reservoir. Find a function W(t) that gives the number of gallons of water in the reservoir.

g) What is the maximum amount of water in the reservoir? The minimum amount?

Maximum amount: _____

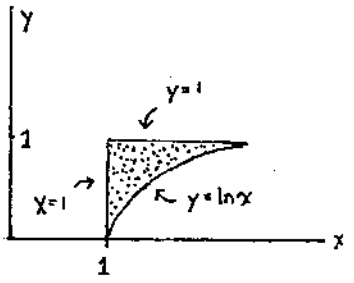
Minimum amount: _____

h) What was the average amount of water in the reservoir in 1994 (between t=0 and t=12)?

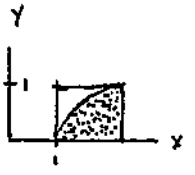
i) The town had successfully fought against the relocation of some heavy industry into the area. In doing so, it argues that the industry would have consumed water at a steady rate of 500 gallons/month. Assuming this is correct, for what values of t, $0 \le t \le 12$, would the amount of water in the reservoir be decreasing if the heavy industry was relocated in the area?

6.(7 points)

a) Find the area bounded by $y = \ln x$, $y = 1$ and $x = 1$. Give an exact answer and show all your work. (Note: we advise you to integrate with respect to y in order to make the integration simplest.)



b) If we were interested in an *approximation* of the area described in part a) we could approach the problem by subtracting from the rectangular area the shaded area under the graph of lnx where we approximate $\int_1^e \ln x\, dx$ using a calculator or computer. Using some fixed number of subdivisions to approximate $\int_1^e \ln x\, dx$, a calculator gives a right hand sum, R_n, a lefthand sum, L_n, a trapezodial approximation, T_n, and a midpoint approximation M_n. Please put R_n, L_n, T_n, M_n, and $I = \int_1^e \ln x\, dx$ in ascending order. (Place < signs between them.)



7. (3 points) Suppose that temperature of an object over a period of time is given by $\frac{40\,t}{1+t^4}$ degrees, t measured in weeks. What is the average temperature of the object over the time interval [0,1]? Please give an exact answer.

10. (9 points) A cylindrical chocolate cake is 4 inches high and 10 inches in diameter. The density of chocolate in the cake varies with the height and is given by $\delta(h) = \left(\frac{1}{15} - \frac{h}{60}\right)$ ounces per square inch, where h measures the height above the table, in inches. We want to know many ounces of chocolate were used in the cake.

a) In order to approximate the number of ounces, how would you slice the cake?

b) Give an expression that approximates the number of ounces of chocolate in the ith slice of the cake using the slicing method you described in a).

c) Write a general Riemann sum (assuming n slices) that approximates the number of ounces of chocolate in the cake, and, taking the appropriate limit, write an integral giving the number of ounces.

d) Use your answer to c) to find the number of ounces of chocolate in the cake. (Please give an exact answer and followed by a numerical estimate.)

11. (10 points) Suppose we are making iced coffee and we place a hot cup of coffee in a refrigerator. Let T(t) be the temperature of the coffee at time t, t given in minutes. The refrigerator is kept at 5 degrees Celsius. Newton's Law of Cooling says that the rate of change of difference between the temperature of the coffee and the temperature of the refrigerator is proportional to this temperature difference.

a) Write a differential equation reflecting this situation.

b) Suppose the coffee cools from 95 degrees at $t = 0$ to 50 degrees 10 minutes later. Find T as a function of t and determine how much longer will it take for the coffee to cool to 10 degrees.

c) Graph the solution to the differential equation requested in part a) using the information given in b). Label important points and asymptotes (if any).

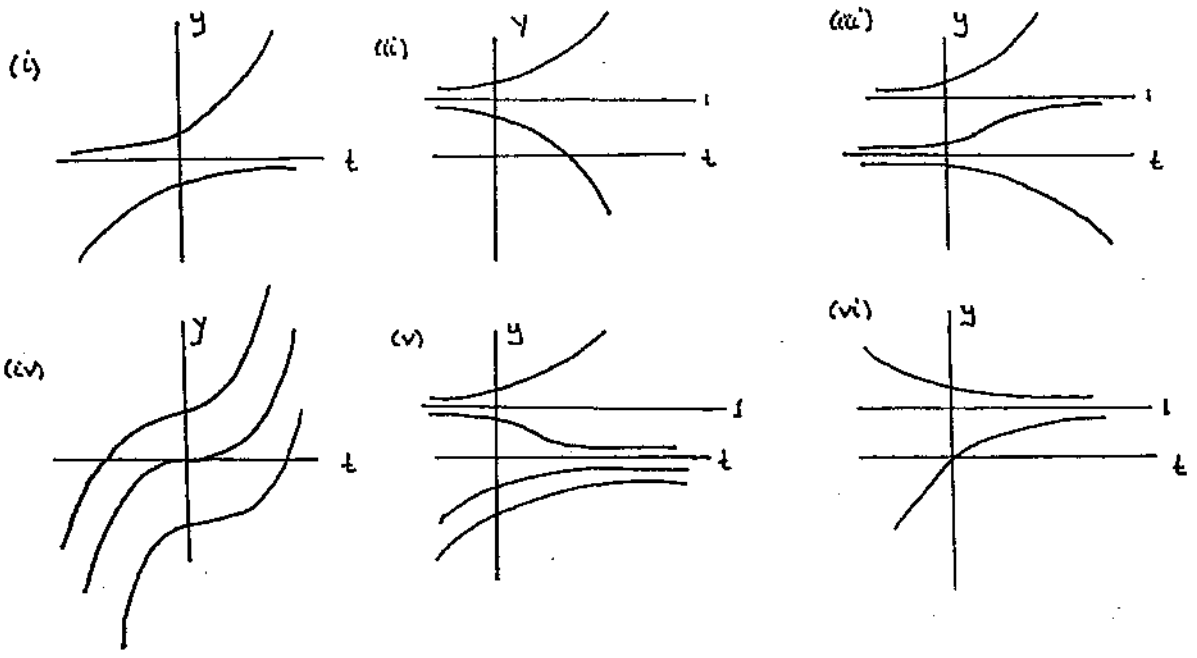
d) What is the rate of change of temperature of the coffee at t=0?

12. (13 points) Differential Equations

i) (4 points) To each differential equation match the graph of some representative solutions:

a) $\frac{dy}{dt} = 3t^2$ b) $\frac{dy}{dt} = y\,(y\text{-}1)$ d) $\frac{dy}{dt} = 3(y\text{-}1)$ $\frac{dy}{dt} = (1\text{-}y)^2\, y$

graph _____ graph _____ graph _____ graph _____



ii) (4 points) Which of the following is a solution to the differential equation $y'' + 4y = 0$. Please show your work.

a) $y = 5\, e^{2t}$

b) $y = 3 \sin (2t)$

c) $y = 2t^2$

d) none of the above

iii) (5 points)

An initial deposit of M_0 dollars is put in a bank account paying interest at a rate of 6% compounded continuously. Money is being withdrawn continuously at a rate of 900 dollars/year.

a) Write a differential equation that reflects this situation. (Let M = M(t) be the amount of money in the account at time t.)

b) What initial deposit would keep the amount of money in the account at a constant level?

8. (5 points) Find the equation of the tangent line to $x \ln y + x^2 y = 2x - y^2 + 1$ at the point (2,1).

9. (6 points) A boat is pulled by a rope towards a dock. The dock is 3 feet above the bow of the boat and the rope is being pulled in (shortened) at a rate of 11 ft/minute. The rope makes an angle of θ with the horizontal (see picture). Is θ increasing or decreasing? Find the rate of change of θ when the length of the rope from the dock to the boat is 5 feet.