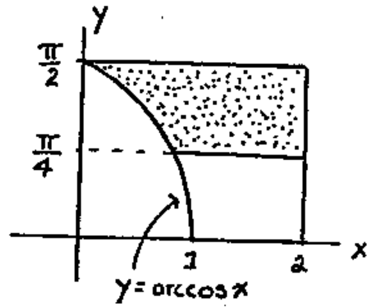
Mathematics Xb

Final Examination

May 17, 1996

1. (10 points) Find the area of the shaded region, the region bounded by $y = \pi/4$, $y = \pi/2$, $x = 2$, and $y = \arccos(x)$. Please give an exact answer. Note: we advise you to integrate with respect to y in order to make the computation simpler.



2. (7 points) The population density in a certain town in Central America varies with the distance from the heart of the central plaza, Plaza Mayor. The population is given by $\partial(x)$ people/ square mile, where x is the distance from the center of town. Write an integral that gives the number of people living more than 10 miles from the center of town but within 20 miles of the center of town.

3. (10 points) Find $\frac{dy}{dx}$:

a) $y = \pi \sin\left(\frac{1}{x}\right) + \pi$

b) $y = \left(\arctan(\sqrt{x})\right)^3$

c) If $y = (x^2+1)^{\sin x}$, find $\frac{dy}{dx}$ and $y'(\pi)$.

4. (10 points) Consider the equation $1 + \cos(xy) + y = xy^2 + 2x$

a) Find dy/dx . (Leave your answer in terms of x and y.)

b) Find the equation of the tangent line at the point (1, 0).

5. (13 points) Compute the following:

a) $\int x \cos(x^2 + 3)\, dx$

c) $\int \frac{e^{1/x}}{x^2}\, dx$

b) $\int \frac{\pi x + \pi}{x^2}\, dx$

d) $\int_0^1 \frac{x}{1 + x^4}\, dx$

e) $\int_{-1}^4 |3x - 6|\, dx$

6. (10 points) The price of a gallon of gas is increasing at a rate of $\ln(0.05x + 1)$ dollars per day.

a) Write an integral that tells us how much the price has increased between $x = 0$ and $x = 4$.

b) Suppose you plan to approximate the value of this integral by chopping the interval [0,4] into n equal pieces and taking left and right hand sums. What is the smallest number of chops you can take such that the difference between the left and right hand sums is less than 0.01?

c) Using your answer to b), determine the increase in the price of a gallon of gas from $x = 0$ to $x = 4$. Please give your answer to the nearest penny.

d) If the price of a gallon of gas is $1.18 at $x = 0$, what is it at $x = 4$?

e) If you average left and right hand sums to approximate the integral in part a), will your answer be larger than the value of that integral, or smaller? Explain your reasoning fully and completely. Please give an argument that does not rely on your calculator.

7. (10 points) A rectangular plot of land is 60 meters long and 30 meters wide. The concentration of a certain mineral in the soil varies with the depth from the surface; the density at depth h is given by $\partial(h) = 0.01\, h \sqrt{h^2 + 1}$ grams/cubic meter. We are interested in finding the number of grams of the mineral contained in a rectangular region 60 meters long, 30 meters and 5 meters deep.

a) Draw a picture of the region, extending to 5 meters under the soil.

b) Write a Riemann sum that approximates the amount of mineral in the soil.

c) Find the exact amount of the mineral in the region (using integration).

8. (10 points) $3,000 is deposited in a bank account with a nominal annual interest rate of 7% rate compounded continuously. Let $M = M(t)$ be the amount in the account at time t.

a) What is the effective annual interest rate?

Suppose money is being withdrawn continuously at a rate of $200 per year.

b) Write a differential equation reflecting the situation.

c) Which one of the following statements is true given the initial conditions established? (Circle the correct answer.)

i) M(t) is increasing at an increasing rate
ii) M(t) is increasing at an decreasing rate
iii) M(t) is decreasing at an increasing rate
iv) M(t) is decreasing at an decreasing rate

d) What is the minimum initial deposit that will support the $200 withdrawal rate?

e) Given that our initial deposit was $3000, what is the maximum withdrawal rate such that the balance does not decrease?

f) Which one of the following is a solution to the differential equation in part a)? You MUST show all your work and reasoning clearly and convincingly.

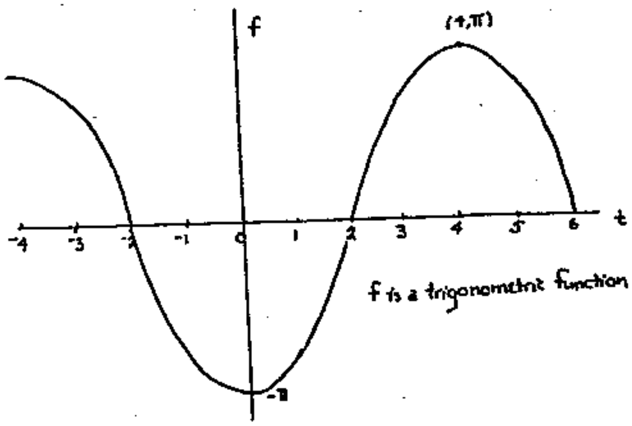
i) $M(t) = Ce^{.07t} - 200t$

ii) $M(t) = Ce^{.07t} + \dfrac{20{,}000}{7}$

iii) $M(t) = Ce^{.07t} - \dfrac{20{,}000}{7}t$

g) What is the particular solution corresponding to the initial deposit of $3000?

9. (10 points) Below is the graph of f(t). Let $A_f(x) = \int_{-2}^{x} f(t)\,dt$.



a) Put the following values in ascending order with < or = signs between them.

$A_f(-4)$, $A_f(-2)$, $A_f(0)$, $A_f(2)$, $A_f(4)$, $A_f(6)$

b) For what values of x between -4 and 6 is A_f increasing?

c) For what values of x between -4 and 6 is A_f concave down?

d) Assuming f is a trigonometric function, give an equation for f.

10. (10 points) Lars the bear dreams that he bumps into a beehive named Carma. The moment Lars bumps into her, Karma sprouts legs and runs away with Lars in pursuit. Carma's velocity is given by $v(t) = 2\cos(t) + t + 3$ and Lars's velocity is given by $v(t) = 2t$.

a) Who is ahead at $t = \pi$? Explain

b) Use your calculator to approximate the time when Carma's lead is largest.

c) Consider the function f(t), the difference between the bear's velocity and Carma's.

$$f(t) = 2t - [\, 2\cos(t) + t + 3 \,]$$

Find and analyze all the critical points of f(t) on the interval $[0,2\pi]$ (including endpoints), and classify them. Distinguish between local maxima and minima and absolute maxima and minima on the interval.

d) At the local maxima and minima of f(t), what can you say about Carma's acceleration as compared with that of Lars?

e) Where on the interval $[0, 2\pi]$ is Carma's lead shrinking fastest?

f) Does Lars ever catch Carma? Explain. (Do not restrict yourself to the interval $[0,2\pi]$.)