Final Examination

Mathematics Xb

May 21, 1999

1. (8 points) Find dy/dx.
   (a) $y = 3x^{x+2}$
   (b) $x^2 + y = \sin(xy)$ Your answer can be in terms of x and y.

2. (13 points) Let $f(x) = e^{-x}\sin x$ on $[0, 3\pi]$.
   (a) Where on $[0, 3\pi]$ is $f(x) = 0$?
   (b) Given the answer to (a) and the fact that f is continuous on $[0, 3\pi]$, which of the following statements *must* be true?
      i. f has no more than two local extrema on $[0, 3\pi]$.
      ii. f has at least three local extrema on $[0, 3\pi]$.
      iii. f has at least four local extrema on $[0, 3\pi]$.
      iv. f has no local extrema on $[0, 3\pi]$.
   (c) Find the exact x-coordinate(s) of all critical points. Do not give numerical approximations.
   (d) Classify each critical point as a local maximum, a local minimum, or neither.
   (e) Find the exact x-coordinate(s) of the global maximum on $[0, 3\pi]$.
   (f) Find the exact x-coordinate(s) of the global minimum on $[0, 3\pi]$.
   (g) Which is larger, $\int_0^{\pi} e^{-x}\sin x\,dx$ or $\int_0^{3\pi} e^{-x}\sin x\,dx$? Explain your reasoning briefly.
   (This part of the problem can be done without having gotten the exact answers to the previous parts.)

3. (9 points) A company opens a bank account with an initial balance of \$30,000. Assume that the account pays 4% interest compounded continuously. The company makes withdrawals at a rate of \$2000 per year.
   (a) Write a differential equation whose solution is $B(t)$, the balance in the account t years after the account is opened.
   (b) Will the bank account eventually be depleted?
   (c) Solve for $B(t)$. Your answer should contain no unknown constants.
   (d) If the company wanted the balance in the account to remain constant, what should the withdrawal rate be?

4. (8 points)
   (a) Is $y = 3\cos 5t$ a solution to the differential equation $y'' + 25y = 0$? Show your reasoning.
   (b) Consider the differential equation $dy/dt = y^2(y - 4)$. On the axes given, do the following. (You do NOT need to worry about the exact position of inflection points.)
      i. Sketch all equilibrium solutions and label each as stable, unstable, or semi-stable.
      ii. Sketch the solution passing through $y(0) = -1$.
      iii. Sketch the solution passing through $y(0) = 3.5$.
      iv. Sketch the solution passing through $y(0) = 5$.

5. (7 points) A farm machine is pouring oats at a rate of 6 cubic feet per minute into a conical pile whose radius is always equal to its height. At what rate is the radius changing when the pile is 10 feet tall? (The volume of a cone is $\frac{1}{3}\pi r^2 h$.)

6. (10 points) Evaluate. Write your answer in simplest form.
   (a) $\int x^2(x^3 + 5)^{10}\,dx$

(b) $\int \frac{\ln x}{2x}\,dx$

(c) $\int_0^{\pi/2} 2\sqrt{\sin x}\cos x\,dx$ Give an exact simplified answer, not a numerical approximation.

7. (9 points) Veronique is saving money for a trip to France. She plans to set aside \$40 each month in an account paying 6% interest compounded monthly (i.e. paying 0.5% interest every month). She makes her first deposit today. How much money will she have immediately after her 20th payment? Give an exact answer in closed form and an approximation correct to the nearest penny.

8. (4 points) Determine whether each of the following series converges or diverges. Explain your criterion. If the series converges, tell us what it converges to.

   (a) $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} \ldots + (-1)^n \frac{1}{3^n} + \ldots$

   (b) $e^{-1} + e^{-2} + e^{-3} + e^{-4} + \ldots + e^{-n} + \ldots$

9. (3 points)

   For each differential equation below sketch graphs of representative solutions.

   a) $\frac{dy}{dt} = y(1-y)$ b) $\frac{dy}{dt} = \cos t$ c) $\frac{dy}{dt} = \cos y$

10. (10 points)

    (a) A magician is performing in front of a crowd in Harvard Square. He clears all people from an area within 4 feet of himself, creating a circular performance area. The density of people in the crowd is given by $\rho(x)$ people/square foot where x is the distance (in feet) from the center of his stage, $4 \le x \le 8$.

    Write an integral that gives the number of people within 8 feet of the magician.

    (b) A hemispherical stone of radius 10 cm is cut from a sedimentary rock sample, where the base of the hemisphere is cut from the bottom of the sample. The density of a particular mineral in the sample is given by $\delta(x)$ milligrams per cubic centimeter where x is the height above the bottom of the sample.

    i. In order to approximate the amount of this particular mineral in the stone, how should you slice up the hemisphere? Please sketch a picture of the sliced hemisphere and of a representative slice.

    ii. Give an expression that approximates the number of milligrams of the mineral in the ith slice using the slicing method you described in (i).

    iii. Write a general Riemann sum (assuming n slices) that approximates the number of milligrams of the mineral in the hemispherical stone, and, by taking the appropriate limit, write an integral giving the amount of the mineral in the hemispherical stone.

    Riemann Sum: __________ limit of Riemann Sum: __________ Integral: __________

11. (15 points) Water is flowing into a reservoir at a steady rate of 450 gallons per hour. The rate at which water is consumed (taken out of the reservoir) over a 24 hour time period is given by the graph below.

    (a) When is the water level in the reservoir the highest? __________

    (b) When is the water level in the reservoir the lowest? __________

    (c) When is the water level in the reservoir increasing most rapidly? __________

    (d) When is the water level in the reservoir decreasing most rapidly? __________

    (e) Assuming that the water consumption is cylic with a 24-hour cycle, write a trigonometric function $r(t)$ modelling the rate of consumption of water.

    Answer: __________

(f) If this situation continues, will the reservoir run dry? Explain your reasoning.

(g) Suppose that we respresent the amount of water in the reservoir at time $t = 0$ by A_0. Describe in plain English what each of the following expressions represents.

i. $\int_0^5 r(t)\,dt$ ____

ii. $(450)(5) - \int_0^5 r(t)\,dt$ ____

iii. $A_0 + (450)(5) - \int_0^5 r(t)\,dt$ ____

12. (4 points) Below is the graph of f. We are interested in the relationship between $\int_0^5 f(x)\,dx$ and various numerical approximations to this definite integral.

We partition the interval $[0, 5]$ into 100 equal pieces, each of length $\Delta x = \frac{5}{100} = 0.05$.

$$\text{Let } x_k = k\Delta x \text{ for } k = 0, 1, 2, \ldots, 100.$$

Put the following expressions in ascending order (with $<$ signs between them). Explain your reasoning.

$$A = \sum_{k=0}^{99} f(x_k)\,\Delta x \qquad B = \sum_{k=1}^{100} f(x_k)\,\Delta x \qquad C = \sum_{k=1}^{100} f\left(\tfrac{1}{2}(x_{k-1} + x_k)\right)\Delta x$$

$$D = \frac{A + B}{2} \qquad E = \int_0^5 f(x)\,dx$$