Solutions to Final Exam
Math Xb 1995

1a) $y = x\ln\left(\frac{x}{\sqrt{x}}\right) = x\ln\sqrt{x} = x\ln x^{1/2} = \frac{1}{2}x\ln x$

$\frac{dy}{dx} = x\frac{1}{2x} + \frac{1}{2}\ln x = \frac{1}{2}(1+\ln x)$

b) $y = 2x^{x+1}$

$\ln y = \ln 2 + (x+1)\ln x$

$\frac{1}{y}\frac{dy}{dx} = \frac{(x+1)}{x} + \ln x$

$\frac{dy}{dx} = y\left[1+\frac{1}{x}+\ln x\right]$

$\frac{dy}{dx} = 2x^{x+1}\left[1+\frac{1}{x}+\ln x\right]$

c) $y = [\sin(x^2)]^5 + 3^x$

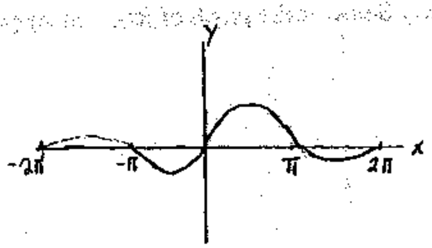
$\frac{dy}{dx} = 5[\sin(x^2)]^4\cos(x^2)\cdot 2x + 3^x\ln 3$

$\frac{dy}{dx} = 10x\sin^4(x^2)\cos(x^2) + 3^x\ln 3$

d) $\frac{d}{dx}\int_3^x \frac{\sin t}{t}\,dt = \frac{\sin x}{x}$ by the Fundamental Th^m of Calculus

2. A: (ii) B: (iv) C: (v) D: (iii) E: (i)

3. Look at the graph of $e^{-x}\sin x$



$D < C < B < A$

(C should have read $\int_{-2\pi}^{0} e^{-x}\sin x\,dx$, but the answer holds regardless.)

4. a) $\int \frac{p}{p+2}\,dp$ let $u = p+2$, $p = u-2$, $dp = du$

$\int \frac{u-2}{u}\,du$

$\int\left(1-\frac{2}{u}\right)du = u - 2\ln|u| + C$
$= p+2 - 2\ln|p+2| + C$

b) $\int_{-3}^{1}|x+2|\,dx$

= Area of Δs drawn

$= \frac{1}{2}(1)(1) + \frac{1}{2}(3)(3) = \frac{1}{2} + \frac{9}{2} = 5$

c) $\int \frac{(\ln x)^2}{x}\,dx$ $u = \ln x$, $du = \frac{1}{x}dx$

$\int u^2\,du =$

$\frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C$

d) $\int \frac{e^x}{(7+e^x)^3}\,dx$ $u = e^x + 7$, $du = e^x dx$

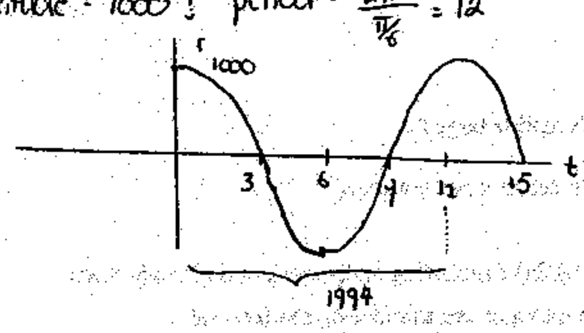
$\int \frac{du}{u^3} = \frac{u^{-2}}{-2} + C = \frac{-1}{2(7+e^x)^2} + C$

Note - you could have used $u = e^x$ and then say $v = u+7$.

5. $r(t) = 1000\cos\left(\frac{\pi}{6}t\right)$ gallons/month

Amplitude = 1000 ; period = $\frac{2\pi}{\pi/6} = 12$

a)



Note: it changes by $\int_6^9 r(t)\,dt = -\frac{6000}{\pi}$, so it is lower by $\frac{6000}{\pi}$.

b) lowest at $t = 9$, Oct. 1st

c) July 1st ($t = 6$) — Note: Question should say July 1, 1994.

d) It is the same

e) It is lower by $\left|\int_6^9 1000\cos\left(\frac{\pi}{6}t\right)dt\right|$

$= \left|1000\int_6^9\cos\left(\frac{\pi}{6}t\right)dt\right| = \left|1000\,\frac{\sin\left(\frac{\pi}{6}t\right)}{\pi/6}\right|$

$= \left|1000\cdot\frac{6}{\pi}\left[\sin\left(\frac{3\pi}{2}\right) - \sin\pi\right]\right| = \left|-\frac{6000}{\pi}\right|$

lower by $\frac{6000}{\pi}$

5. f) $r(t) = 1000\cos\left(\frac{\pi}{6}t\right) = \frac{dW}{dt}$

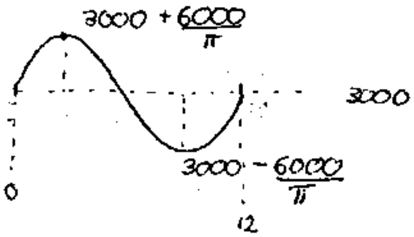
$W(t) = \int 1000\cos\left(\frac{\pi}{6}t\right)dt = 1000\,\frac{\sin\frac{\pi}{6}t}{\frac{\pi}{6}} + C = \frac{6000}{\pi}\sin\left(\frac{\pi}{6}t\right) + C$

$W(0) = 3000 = \frac{6000}{\pi}\sin(0) + C = 0 + C \Rightarrow C = 3000$

$W(t) = 3000 + \underbrace{\frac{6000}{\pi}}_{\text{Amplitude} = \frac{6000}{\pi}}\sin\left(\frac{\pi}{6}t\right)$

g) Max. value of $W(t)$ is $3000 + \frac{6000}{\pi}$

Minimum value of W is $3000 - \frac{6000}{\pi}$



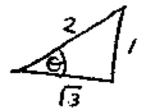
h) Ave. value of $W = \frac{1}{12}\int_0^{12}\left[3000 + \frac{6000}{\pi}\sin\left(\frac{\pi}{6}t\right)\right]dt$: just by looking at the graph we see the average value of W is 3000.

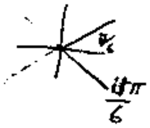
i) The new rate $r(t)$ would be $1000\cos\left(\frac{\pi}{6}t\right) - 500$. The question is when is $r(t)$ negative. To answer this, set $r(t) = 0$.

$1000\cos\left(\frac{\pi}{6}t\right) - 500 = 0$

$\cos\left(\frac{\pi}{6}t\right) = \frac{500}{1000} = \frac{1}{2}$

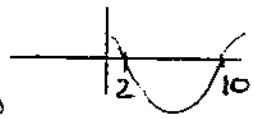
$\cos\left(\frac{\pi}{6}t\right) = \frac{1}{2}$ $\qquad \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}, \ldots$





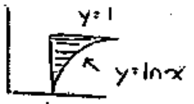
so $\frac{\pi}{6}t = \frac{\pi}{3}$ or $\frac{\pi}{6}t = \frac{5\pi}{3}$

$t = 2$ or $t = 10$



The amount of water would be decreasing from $t = 2$ to $t = 10$

6. a.



$\Leftrightarrow e^y = x$ $\qquad$ Area $= \int_0^1 (e^y - 1)\,dy = (e^y - y)\Big|_0^1 = (e-1) - (1) = e - 2$

b.

Since $\ln x$ is increasing, $L_n < R_n$ $\quad (L_n < I < R_n)$

Trapezoidal & midpt. rules are more accurate approximations of I.

Since $\ln x$ is concave down, $T_n < I$ and $M_n > I$ so

$L_n < T_n < I < M_n < R_n$ $\qquad$ (1995-6: omit M_n & order the others!)

7. $\frac{1}{1}\int_0^1 \frac{40t}{1+t^4}\,dt = 40\int_0^1 \frac{t}{1+t^4}\,dt$ $\qquad$ let $u = t^2$, $du = 2t\,dt$ $\qquad t\,dt = \frac{du}{2}$

$\frac{40}{2}\int_{t=0}^{t=1} \frac{1\,du}{1+u^2} = 20\arctan u\,\Big|_{t=0}^{t=1} = 20\arctan t^2\Big|_0^1 = 20[\arctan 1 - \arctan 0] = \frac{20\pi}{4} = 5\pi$

(Recall: Ave val of f on $[a,b]$ $= \frac{1}{b-a}\int_a^b f(x)\,dx$)

PART II.

8. Differentiate implicitly: $x\frac{1}{y}\frac{dy}{dx} + \ln y + 2xy + x^2\frac{dy}{dx} = 2 - 2y\frac{dy}{dx}$. $\qquad$ solve for $\frac{dy}{dx}$

$\frac{x}{y}\frac{dy}{dx} + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = -\ln y - 2xy + 2$

$$\frac{dy}{dx}\left(\frac{x}{y}+x^2+2y\right) = -\ln y - 2xy + 2$$

$$\frac{dy}{dx} = \frac{-\ln y - 2xy + 2}{\frac{x}{y}+x^2+2y} \quad \text{at } \begin{matrix}x=2\\ y=1\end{matrix} \Rightarrow \frac{dy}{dx} = \frac{-\ln(1)-4+2}{2+4+2} = \frac{-2}{8} = -\frac{1}{4}$$

Eq'n of tangent line: slope $-\frac{1}{4}$ pt. $(2,1)$ $\qquad y = -\frac{1}{4}x + b$

$1 = -\frac{1}{2} + b \Rightarrow b = \frac{3}{2}$

$\boxed{y = -\frac{1}{4}x + \frac{3}{2}}$

9. See end of next page.

10. a)  slice into disks parallel to the table.

b. # of ounces of chocolate $\approx \left(\frac{1}{15} - \frac{h_i}{60}\right)$ ounces/cubic inch (typo) $\cdot \underbrace{(5)^2\pi\,\Delta h}_{\text{vol. of a slice : } \pi r^2 \Delta h}$ cubic inches

so $\approx \left(\frac{1}{15} - \frac{h_i}{60}\right) 25\pi\,\Delta h$ ounces in the i^{th} slice

c. $\sum_{i=1}^{n} \left(\frac{1}{15} - \frac{h_i}{60}\right) 25\pi\,\Delta h \approx$ total amt of chocolate.

Total chocolate $= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{15} - \frac{h_i}{60}\right) 25\pi\,\Delta h = \int_0^4 \left(\frac{1}{15} - \frac{h}{60}\right) 25\pi\, dh$

d) $\boxed{25\pi \int_0^4 \left(\frac{1}{15} - \frac{h}{60}\right) dh} = \left[\left(\frac{1}{15}h - \frac{h^2}{120}\right)\Big|_0^4\right] 25\pi = \left(\frac{4}{15} - \frac{16}{120}\right) 25\pi = \left(\frac{8}{30} - \frac{4}{30}\right) 25\pi = \frac{50\pi}{15}$

$\boxed{= \frac{10}{3}\pi \text{ ounces} \approx 10.47 \text{ ounces}}$

11. a) $\frac{d(T-5)}{dt} = k(T-5)$ or $\frac{dT}{dt} = k(T-5)$

b) let $w = T-5$ $\quad \frac{dw}{dt} = kw \Rightarrow w = Ce^{kt}$ so $T-5 = Ce^{kt} \Rightarrow T = 5 + Ce^{kt}$

use the info given to find C & K.

when $t=0$ $T=95$ so $95 = 5 + Ce^0 \Rightarrow 95 = 5 + C \Rightarrow C = 90$ so $T = 5 + 90e^{kt}$

when $t=10$ $T=50$ so $50 = 5 + 90e^{k\cdot 10}$

$45 = 90e^{10k}$

$\frac{1}{2} = e^{10k}$

$\ln(\frac{1}{2}) = 10k \Rightarrow k = \frac{\ln(\frac{1}{2})}{10} = \frac{-\ln 2}{10}$ ← either approximate numerically or -

$T = 5 + 90e^{(-\frac{1}{10}\ln 2)t}$ $\qquad$ where $e^{(-\frac{1}{10}\ln 2)t} = \left(e^{\ln 2^{-1/10}}\right)^t = \left(2^{-1/10}\right)^t = 2^{-t/10}$

so $\boxed{T = 5 + 90\left(2^{-t/10}\right)}$ or $\boxed{T = 5 + 90e^{-.069t}}$

Now set $T = 10$ $\qquad 10 = 5 + 90\,2^{-t/10}$

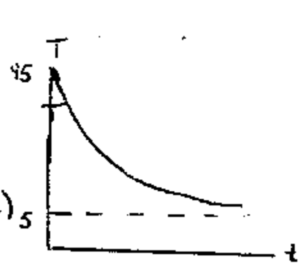
$5 = 90 \cdot 2^{-t/10}$

$\frac{1}{18} = 2^{-t/10}$

$\ln(\frac{1}{18}) = -\frac{t}{10}\ln 2$

$\frac{\ln(\frac{1}{18})}{\ln 2} = -\frac{t}{10}$

$t = \frac{-10\ln(\frac{1}{18})}{\ln 2} = 41.7$ minutes

c) 

d) $\frac{d(T-5)}{dt} = k(T-5) \Rightarrow \frac{dT}{dt} = k(T-5)$ where $k = \frac{-\ln 2}{10}$. so at $t=0$ $\frac{dT}{dt} = \frac{-\ln 2}{10}(95-5)$

Note: you could compute $\frac{dT}{dt}$ from scratch from your ans. to b) $\qquad = -9\ln 2$ deg/min.

12. (Note: in 1995-6 a) was not treated extensively)

a) (iv) $\frac{dy}{dt} = 3t^2$ b) (V) $\frac{dy}{dt} = y(y-1)$ c) (ii) $\frac{dy}{dt} = 3(y-1)$ d) (iii) $\frac{dy}{dt} = (1-y)^2 \cdot y$

ii) Try the solns:

a) does $y = 5e^{2t}$ work?

$y' = 10e^{2t}$

$y'' = 20e^{2t}$

$y'' + 4y \stackrel{?}{=} 0$

$20e^{2t} + 4(5e^{2t}) \stackrel{?}{=} 0$

$40e^{2t} \neq 0$ No!

b) does $y = 3\sin(2t)$ work?

$y' = 6\cos(2t)$

$y'' = -12\sin(2t)$

$y'' + 4y \stackrel{?}{=} 0$

$-12\sin(2t) + 4(3\sin(2t)) \stackrel{?}{=} 0$

$-12\sin(2t) + 12\sin(2t) = 0$ ✓ Yes, it works!

c) doesn't work

$y' = 4t$

$y'' = 4$

$4 + 4(2t^2) \neq 0$

ANSWER: b

iii) a) $\frac{dM}{dt} = .06M - 900$

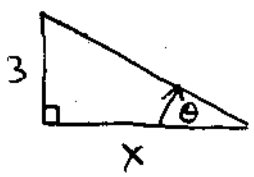
b) $\frac{dM}{dt} = .06M - 900 = 0$

$.06M = 900$

$M = \frac{900}{.06} = 15{,}000$

so an initial deposit of \$15,000 would keep the amount of money in the account at a constant level.

7.



Want: $\frac{d\theta}{dt}$ Know: $\frac{dx}{dt}$

So, relate θ and x, then take $\frac{d}{dt}$.

$\tan\theta = \frac{3}{x}$

$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{3}{x}$

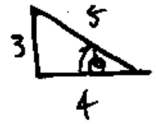
$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{-3}{x^2} \cdot \frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{-3}{x^2} \cdot \cos^2\theta \cdot \frac{dx}{dt}$

$= \frac{-3}{16} \cdot \frac{16}{25} \cdot (-11)$

$= \frac{33}{25} \frac{\text{rad}}{\text{min}}$ (θ is increasing, as we expect.)

At moment in question:



$x = 4$

$\cos\theta = \frac{4}{5}$