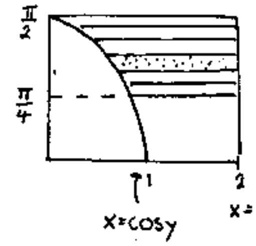
May 17, 1996

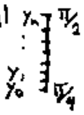
1.



First write the right & left hand boundary in the form $x = f(y)$

$y = \arccos x \Rightarrow \cos y = x$

Partition the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ into n equal pieces as shown. Each piece is of length Δy



Area of i^{th} slice $\approx (2-\cos y_i)\Delta y$

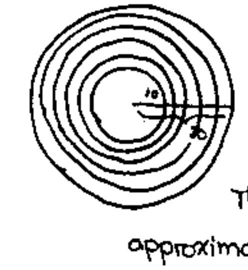
Total area $\approx \sum_{i=1}^{n} (2-\cos y_i)\Delta y$

$$\text{Area} = \int_{\pi/4}^{\pi/2} (2-\cos y)\,dy$$

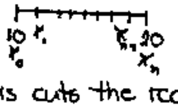
$$= (2y - \sin y)\Big|_{\pi/4}^{\pi/2} = 2\left(\tfrac{\pi}{2}\right) - \sin\tfrac{\pi}{2} - \left[2\tfrac{\pi}{4} - \sin\tfrac{\pi}{4}\right]$$

$$= \pi - 1 - \left[\tfrac{\pi}{2} - \tfrac{\sqrt{2}}{2}\right] = \tfrac{\pi}{2} - \tfrac{2}{2} + \tfrac{\sqrt{2}}{2} = \boxed{\frac{\pi - 2 + \sqrt{2}}{2}}$$

2.



Partition $[10, 20]$ into n equal pieces



, each of length Δx

This cuts the region into concentric rings

in which the density is approximately constant.

Area $\approx 2\pi x_i \Delta x$ $\quad \underbrace{\quad}_{\approx 2\pi x_i} \mid \Delta x$

pop. in i^{th} ring $\approx \underbrace{\partial(x_i)}_{\text{people/sq mi}} \underbrace{2\pi x_i \Delta x}_{\text{sq. miles}}$

Total pop. $\approx \sum_{i=1}^{n} \partial(x_i)\, 2\pi x_i \Delta x$ $\Rightarrow$ taking $\lim_{n\to\infty} \sum_{i=1}^{n} \partial(x_i)\, 2\pi x_i \Delta x$ gives $\int_{10}^{20} \partial(x)\, 2\pi x\, dx$

or $\boxed{2\pi \int_{10}^{20} x\,\partial(x)\,dx}$

3. a) $\frac{dy}{dx} = \pi\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + 0$

$\frac{dy}{dx} = -\frac{\pi}{x^2}\cos\left(\frac{1}{x}\right)$

b) $\frac{dy}{dx} = 3\left[\arctan\sqrt{x}\right]^2 \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2}x^{-1/2}$

$\frac{dy}{dx} = 3\left[\arctan\sqrt{x}\right]^2 \cdot \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}}$

c) let's use logarithmic differentiation

$\ln y = \sin x \ln(x^2+1)$ $\quad$ Now differentiate w/ respect to x. Remember the product rule on the right hand side.

$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln(x^2+1) + \sin x \frac{1}{x^2+1} \cdot 2x$$

$$\frac{dy}{dx} = y\left[\cos x \ln(x^2+1) + \frac{2x\sin x}{x^2+1}\right]$$

$$\frac{dy}{dx} = (x^2+1)^{\sin x}\left[\cos x \ln(x^2+1) + \frac{2x\sin x}{x^2+1}\right]$$

$$y'(\pi) = (\pi^2+1)^{\sin\pi}\left[\cos(\pi)\ln(\pi^2+1) + \frac{2\pi\sin\pi}{\pi^2+1}\right]$$

$$= (\pi^2+1)^0\left[-1\ln(\pi^2+1) + 0\right]$$

$$= -\ln(\pi^2+1)$$

4. a) $1 + \cos(xy) + y = xy^2 + 2x$ $\quad$ differentiate implicitly

$$-\sin(xy)\left[1\cdot y + x\frac{dy}{dx}\right] + \frac{dy}{dx} = 1\cdot y^2 + x\,2y\frac{dy}{dx} + 2$$

$$-y\sin(xy) - x\sin(xy)\frac{dy}{dx} + \frac{dy}{dx} = y^2 + 2xy\frac{dy}{dx} + 2$$

$$\frac{dy}{dx}\left[-x\sin(xy) + 1 - 2xy\right] = y^2 + 2 + y\sin(xy)$$

$$\boxed{\frac{dy}{dx} = \frac{y^2+2+y\sin(xy)}{-x\sin(xy)+1-2xy}}$$

b) Eq'n of tangent line at $(1,0)$.
We have a pt - we need the slope of the tangent at $(1,0)$

$$m_{tan} = \frac{0+2+0}{-\sin 0 + 1 - 0} = \frac{2}{1} = 2$$

$y = 2x + b$ $\qquad 0 = 2 + b \Rightarrow b = -2$

$\boxed{y = 2x - 2}$

Alternatively, $2 = \frac{y-0}{x-1} \Rightarrow 2(x-1) = y$

5. a) $\int x\cos(x^2+3)\,dx$

$u = x^2+3$
$du = 2x\,dx$
$\frac{du}{2} = x\,dx$

$\int \cos u \frac{du}{2} = \frac{1}{2}\int \cos u\,du$

$= \frac{1}{2}\sin u + C$

$= \boxed{\frac{1}{2}\sin(x^2+3)+C}$

b) $\int \frac{\pi x+\pi}{x^2}\,dx =$

$\pi\int \frac{x+1}{x^2}\,dx =$

$\pi\int\left(\frac{1}{x} + x^{-2}\right)dx =$

$\pi\left[\ln|x| + \frac{x^{-1}}{-1}\right] + C =$

$\boxed{\pi\left[\ln|x| - \frac{1}{x}\right] + C}$

c) $\int \frac{e^{1/x}}{x^2}\,dx =$

$\int e^{1/x}\,\frac{1}{x^2}\,dx$ let $u = \frac{1}{x}$
$du = -\frac{1}{x^2}\,dx$
$-du = \frac{1}{x^2}\,dx$

$\int e^u(-du) =$

$-\int e^u\,du = -e^u + C =$

$\boxed{-e^{1/x}+C}$

d) $\int_0^1 \frac{x}{1+x^4}\,dx$

let $u = x^2$
$du = 2x\,dx$
$\frac{du}{2} = x\,dx$

when $x=0$, $u=0$
when $x=1$, $u=1$

$\int_0^1 \frac{1}{1+u^2}\frac{du}{2} =$

$\frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du =$

$\frac{1}{2}\arctan u\Big|_0^1 =$

$\frac{1}{2}[\arctan 1 - \arctan 0]$

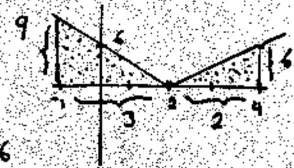
$\frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{1}{2}\frac{\pi}{4}$

$= \boxed{\frac{\pi}{8}}$

e) $\int_{-1}^{4} |3x-6|\,dx$ Caution: resist the temptation to ignore the absolute value signs. You must break this integral up into pieces so that in each piece the sign of the integrand doesn't change. Looking at it graphically is bound to be useful.

$\int_{-1}^{4}|3x-6|\,dx = \frac{1}{2}(9)(3) + \frac{1}{2}(2)(6)$

$= \frac{27}{2} + 6 = \boxed{\frac{39}{2}}$ DONE!



Alternatively: for $x<2$ $|3x-6| = -3x+6$
for $x\geq 2$ $|3x-6| = 3x-6$

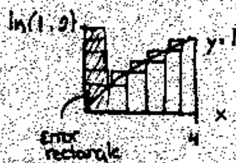
so $\int_{-1}^{4}|3x-6|\,dx = \int_{-1}^{2}|3x-6|\,dx + \int_2^4 |3x-6|\,dx$

$= \int_{-1}^{2}(-3x+6)\,dx + \int_2^4(3x-6)\,dx$

$= \left(\frac{-3x^2}{2}+6x\right)\Big|_{-1}^{2} + \left(\frac{3x^2}{2}-6x\right)\Big|_2^4 = \cdots = \boxed{\frac{39}{2}}$

6. a) $\int_0^4 \ln(.05x+1)\,dx$

b) $\ln(.05x+1)$ is increasing on $[0,4]$. How do we know? $\frac{d}{dx}(\ln(.05x+1)) = \frac{.05}{.05x+1}$ is positive for $x\in[0,4]$



$R_n - L_n = [\ln(1.2) - 0]\frac{4}{n} = \frac{4\ln 1.2}{n} < \frac{1}{100}$

$n > (4\ln 1.2)\,100$
$n > 72.9\ldots$ so $n=73$ is the smallest partition.

c) Approximate $\int_0^4 \ln(.05x+1)\,dx$ using left + right sums

$L_{73} = .37072\ldots$ $R_{73} = .38071\ldots$ The price increase is somewhere between, but to get the answer to the "nearest penny" we'd need more subdivisions. $n=600$ gives 38¢ to the nearest penny.

d) $1.18 + $.38 = $1.56

e) If you average the left and right hand sums you've got the trapezoidal rule. Let's look at the concavity of $f(x) = \ln(.05x+1)$

$f'(x) = \frac{.05}{.05x+1}$ $f''(x) = -.05(.05x+1)^{-2}(.05) = \frac{-(.05)^2}{(.05x+1)^2} < 0$ so f is concave down.

The trapezoidal rule will be smaller than the actual value of the integral

7. The key point here is that density varies with depth so we've got to chop into slabs $60 \times 30 \times \Delta h$

Partition $[0,5]$ into n equal pieces. $h_0 \ldots h_n$ $\quad \Delta h = \frac{5}{n}$

# of grams of mineral in the i^{th} slab $\approx \underbrace{.01 h_i \sqrt{h_i^2+1}}_{\text{density}} \;\; \underbrace{30 \cdot 60 \cdot \Delta h}_{\text{vol. of a slice}}$

Amt. of mineral in the soil $\approx \sum_{i=1}^{n} (.01 h_i \sqrt{h_i^2+1}) \cdot 1800 \Delta h$

Amt $= \lim_{n \to \infty} \sum_{i=1}^{n} (.01 h_i \sqrt{h_i^2+1}) \cdot 1800 \Delta h = \int_0^5 .01 h \sqrt{h^2+1} \cdot 1800\, dh$

$= 18 \int_0^5 h\sqrt{h^2+1}\, dh$

$18 \int_0^5 h\sqrt{h^2+1}\, dh = 18 \int_1^{26} u^{1/2} \frac{du}{2}$

$u = h^2+1$
$du = 2h\,dh$
$\frac{du}{2} = h\,dh$
when $h = 0$, $u = 1$
$h = 5$, $u = 26$

$= 9 \int_1^{26} u^{1/2}\, du$

$= 9 \cdot \frac{2}{3} u^{3/2} \Big|_1^{26} = 6u^{3/2}\Big|_1^{26} = 6(26)^{3/2} - 6$ grams

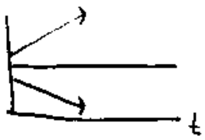
8. a) $M = M_0 e^{.07t} = M_0 (e^{.07})^t \approx M_0 (1.0725)^t$ so the effective interest rate is $\approx 7.25\%$ (a bit over 7%, as expected.)

b) $\frac{dM}{dt} = .07M - 200$

c) If $M = 3000$ at $t=0$ then $\frac{dM}{dt} = (3000 \cdot .07) - 200 = 210 - 200 = 10$ at $t = 0$, so M increases & hence $\frac{dM}{dt}$ increases $\Rightarrow \boxed{i}$

Alternative approach

$\frac{dM}{dt} = 0$ at $M \approx 2857.14\ldots$



$\frac{d^2M}{dt^2} = .07M \frac{dM}{dt} = .07M(.07M - 200)$

so $\frac{d^2M}{dt^2} > 0$ when $M > 2857.14\ldots$

Answer $\boxed{(i)}$

M is increasing at an increasing rate.

d) \$2857.15 (round up - otherwise you'll eventually have no \$.)

e) Solve $\frac{dm}{dt} = 0$

$\Rightarrow m = \frac{200}{.07}$ dollars.

f) $\frac{dM}{dt} = .07M - 200$

Try i) If $M = Ce^{.07t} - 200t$

$\frac{dM}{dt} = .07Ce^{.07t} - 200$

$\frac{dM}{dt} \stackrel{?}{=} .07M - 200$

$[.07Ce^{.07t} - 200] \stackrel{?}{=} .07[Ce^{.07t} - 200t] - 200$

No - Not equal

Try ii) If $M = Ce^{.07t} + \frac{20,000}{7}$

$\frac{dM}{dt} = .07Ce^{.07t}$

$\frac{dM}{dt} \stackrel{?}{=} .07M - 200$

$.07Ce^{.07t} \stackrel{?}{=} .07[Ce^{.07t} + \frac{20000}{7}] - 200$

$.07Ce^{.07t} \stackrel{?}{=} .07Ce^{.07t} + 200 - 200$

$.07Ce^{.07t} = .07Ce^{.07t}$

Equal

So $M = Ce^{.07t} + \frac{20000}{7}$ is a sol'n.

g) When $t = 0$, $M = 3000$ so $\quad 3000 = Ce^0 + \frac{20,000}{7}$

$C = 3000 - \frac{20,000}{7} \quad \Rightarrow \quad M = (3000 - \frac{20,000}{7})e^{.07t} + \frac{20000}{7}$

9. a) 1^{st} group $\begin{cases} \text{neg} & A_f(0),\ A_f(2),\ A_f(4),\ A_f(-4) \\ \text{pos.} & - \\ \text{zero} & A_f(-2),\ A_f(6) \end{cases}$

$A_f(-4) = \int_{-2}^{-4} f(t)\,dt = -\underbrace{\int_{-4}^{-2} f(t)\,dt}_{\text{positive}}$ $= (-1)$ positive $\Rightarrow$ neg.

$A_f(2) < \underbrace{A_f(0) = A_f(4) = A_f(-4)}_{\text{equal}} < \underbrace{A_f(-2) = A_f(6)}_{\text{equal}}$

b) A_f is increasing where f is positive, since $\frac{d}{dx} A_f = f(x)$
$[-4,-2),\ (2,6)$

c) A_f is concave down where f is decreasing
$(-4,0)\ \ (4,6)$

d) f is a cosine curve, flipped, with amplitude π & period 8 :
$f(t) = -\pi\cos(\frac{\pi}{4}t)$

$f(t) = -\pi\cos Bt$ where $\frac{2\pi}{B} = 8$
$B = \frac{\pi}{4}$

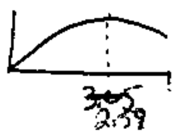
10. a) Compare $\int_0^{\pi}(2\cos t + t + 3)\,dt$ with $\int_0^{\pi} 2t\,dt$

$2\sin t + \frac{t^2}{2} + 3t \Big|_0^{\pi}$ vs. $t^2\Big|_0^{\pi}$

$\frac{\pi^2}{2} + 3\pi = \frac{\pi^2 + 6\pi}{2} \approx 14.359$ vs. $\pi^2 \approx 9.869$

Distance travelled by Carma — Distance travelled by Lars — so Carma is ahead.

b) Compare $\underbrace{2\sin t + \frac{t^2}{2} + 3t}_{C(t) = \text{dist. covered by Carma}}$ with $\underbrace{t^2}_{L(t) = \text{dist. covered by Lars.}}$ for $t > 0$. So let's look at $C(t) - L(t)$

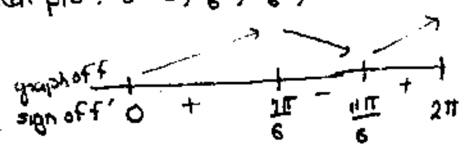


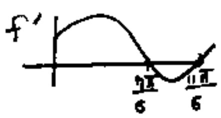
Carma's lead is largest when $t \approx$ ~~3.05~~ 2.5

c) $f'(t) = 2 - [-2\sin t + 1] = 2\sin t + 1$

critical pts: $\begin{cases} f' = 0 \\ f' \text{ undef.} \\ \text{endpts} \end{cases}$ $f' = 0 \Rightarrow 2\sin t = -1$
$\sin t = -\frac{1}{2}$
$t = \frac{7\pi}{6},\ \frac{11\pi}{6}$

critical pts : $t = 0, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi$





local max at $\frac{7\pi}{6}$: also absolute max.
local min at $\frac{11\pi}{6}$
absolute min at $t = 0$

d) at $t = \frac{7\pi}{6}$ Lars' accel. is 2 and Carma's is $-2\sin t + 1\Big|_{t=\frac{7\pi}{6}} = -2(-\frac{1}{2}) + 1 = 2$
They have the same acceleration

at $t = \frac{11\pi}{6}$ Lars's accel is 2 & Carma's is 2 as well. This makes sense since the difference in their accelerations corresponds to the zeros of f' above

e) Where (the bear's velocity minus Carma's velocity) is max – i.e. the max of $f(t)$ on $[0, 2\pi]$
this is at $t = \frac{7\pi}{6}$.

f) Yes. You can look at the graphs of $C(t)$ & $L(t)$ (the distances covered by Carma + Lars at time t.) The graphs intersect. Or, look at $L(t) - C(t) = t^2/2 - 3t - 2\sin t$. This is eventually positive. ($t^2/2$ dominates the expression.) this happens at $\approx 5.4 = t$