Answers to the Second Exam : April 1995

a) $f'(x) = \frac{1}{3}(3x^2 \sin 5x + x^3 \cos 5x \cdot 5) = \frac{1}{3}(3x^2 \sin 5x + 5x^3 \cos 5x)$ [basically product rule]

b) $f'(x) = 3\tan^2\sqrt{x} \cdot \sec^2\sqrt{x} \cdot \frac{1}{2\sqrt{x}}$ [basically [MESS]^3]

c) $f'(x) = 2 \cdot \frac{1}{1+(x^5)^2} \cdot 5x^4 = \frac{10x^4}{1+x^{10}}$ [basically arctan (MESS)]

2. $F(x) = \int_0^x f(t)\,dt \qquad F'(x) = f(x)$

a) F(x) is increasing where its derivative is positive, i.e. where f is positive : (0,8)

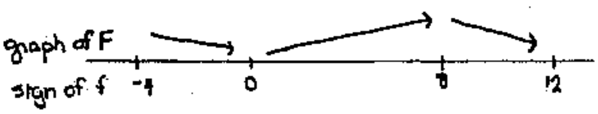
b) F(x) is decreasing where its deriv. is negative, i.e. where f is negative : (-4,0), (8,12)

c) F(x) is concave up where its deriv. is increasing, i.e. where f is increasing : (-2,4), (10,12)

d) $\frac{dF}{dx} = 0 \Leftrightarrow f = 0 : \; x=-4,\; x=0,\; x=8,\; x=12$

e) $F(-2) = \int_0^{-2} f(t)\,dt = -\int_{-2}^{0} f(t)\,dt = -[\text{'neg' area of a quarter of a circle of radius } 2] = -(-\frac{1}{4}\pi(2)^2) = -(-\pi) = \pi$

f)



The max is either at -4 or 8
Let's compare values

$F(-4) = \int_0^{-4} f(t)\,dt = -\int_{-4}^{0} f(t)\,dt = -[-\frac{1}{2}\pi(2)^2] = -[-2\pi] = 2\pi$

$F(8) = \int_0^8 f(t)\,dt = \frac{1}{2}(8)(4) = 16 \Rightarrow$ Max of 16 at $x=8$.

g) candidates for min are $x=0$ & $x=12$

$F(0) = 0 \leftarrow$ Min of 0 at $x=0$

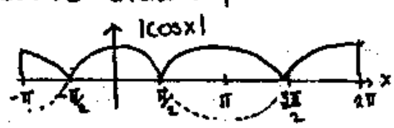
$F(12) = \int_0^{12} f(t)\,dt = 16 - \frac{1}{2}(4)(1) = 16-2 = 14$

h) Net change from 4 to 12 $= \int_4^{12} f(t)\,dt =$ signed area from 4 to 12 $= \frac{1}{2}(4)(4) - \frac{1}{2}(1)\cdot 4 = 8-2 = 6$: 6 gallons

3. a) positive - since the integrand is positive on [1,2] : $\int_1^2 \frac{1}{t^2} + \frac{1}{t}\,dt = \int_1^2 t^{-2} + \frac{1}{t}\,dt = -\frac{1}{t}\Big|_1^2 + \ln t\Big|_1^2 = -(\frac{1}{2}-1) + \ln 2 - \ln 1 = \frac{1}{2} + \ln 2$

b) i) positive - since cos x is positive on $[0, \pi/2]$: $\int_0^{\pi/2} \cos x\,dx = \sin x\Big|_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = \sin\frac{\pi}{2} = 1$

ii) positive : draw a picture $\int_{-\pi}^{2\pi} |\cos x|\,dx = 3\int_{-\pi/2}^{\pi/2} \cos x\,dx = 6\int_0^{\pi/2} \cos x\,dx = 6\cdot(1) = 6$ (from part (i))



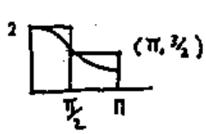
c) positive : $\frac{2}{1+t^2}$ is always positive. $\int_{-1}^{0} \frac{2}{1+t^2}\,dt = 2\arctan t\Big|_{-1}^{0} = 2[\arctan 0 - \arctan(-1)] = 2(0-(-\frac{\pi}{4})) = \frac{\pi}{2}$

4. a) $E < B = D < C < A$

b) $F = \int_{-2}^{0} f(x)\sin x\,dx$. On [-2,0] sin x is negative
Multiplying f (which is positive) by sin x makes the integrand negative - so it's the only negative value.

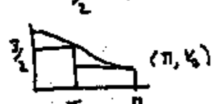
$F < E < B = D < C < A$

c) $\int_0^{\pi} f(x)\,dx$: upper bound :



$2\cdot\frac{\pi}{2} + \frac{3}{2}\cdot\frac{\pi}{2} = (2+\frac{3}{2})(\frac{\pi}{2}) = \frac{7}{2}\cdot\frac{\pi}{2} = \frac{7\pi}{4}$

lower bound :



$\frac{3}{2}\cdot(\frac{\pi}{2}) + \frac{1}{2}(\frac{\pi}{2}) = (\frac{4}{2})\cdot\frac{\pi}{2} = \pi$

d) The area between the 2 curves is defined to be positive - so let's split this into two

shaded area $= \int_0^{\pi/2} [f(x) - g(x)]\,dx + \int_{\pi/2}^{\pi} [g(x) - f(x)]\,dx$

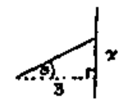
6. a) No, because the rate of admittance > rate arrival before 10

b) The line will be longest at 2:00. At 2:00 there will be 25 people on line. At 10:00 there are 0 people and from 10 to 2, 65 people arrive while 40 people leave. Therefore, at 2 there will be 25 people.

c) Since the line is longest at 2:00, the last person in line at 2:00 must wait the longest. Since there are 25 people and the rate of admittance is 10 pat/hr, this patient will wait 2.5 hrs.

d) The line will disappear ... after 10:00 = # patients admitted after 10:00. This occurs at 5:00 (50 people admitted, 50 people arrived)

e) At 12:00, the line is increasing at a rate of 10 pat/hr (the difference betw rate arrival + rate admit)

7. 

We know: $\frac{dx}{dt} = \cdot 20$ miles/min (assuming the searchlight rotates counterclockwise.)

We want: $\frac{d\theta}{dt}$.

$\tan\theta = \frac{x}{3}$

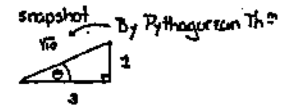
$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{x}{3}$

$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{3}\frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{1}{3}\cos^2\theta \frac{dx}{dt}$

$= \frac{1}{3}\left(\frac{3}{\sqrt{10}}\right)^2 \cdot (\cdot 20)$

$= \frac{3}{10} \cdot 20 = 6 \Rightarrow \frac{d\theta}{dt} = 6$ radians/minute. Now convert rad/min → rev/min



$6\ \frac{\text{radians}}{\text{min}} \cdot \frac{1}{2\pi}\ \frac{\text{revol'n}}{\text{radians}} = \frac{6}{2\pi}\ \frac{\text{revol'ns}}{\text{min}} = \boxed{\frac{3}{\pi}\ \frac{\text{revolutions}}{\text{minute}}}$ (so a bit under 1 rev'n/min.)

1 rev. = 2π radians so this is 1

8. at 7:00, 100 words on paper

lower bd. at 12:00, # words ≥ 100 + 50(2) + 60(2) + 100(.5) + 120(.5) = 430

Lower Bound = 430

upper bd. at 12:00, # words ≤ 100 + 60(2) + 100(2) + 120(.5) + 140(.5) = 550