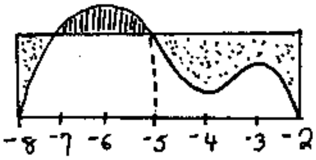
Annotated Solutions to the Second Exam
X6: April 20^{th}, 1998

#1. a) $B < D < E < C < A$

b) ave. value of g on $[-8,-2]$ $= \dfrac{\int_{-8}^{-2} g(w)\,dw}{6}$



if $g_{AVE} = g(-5)$ then the area under the rectangle $[6]\cdot[g(-5)]$ would be equal to $\int_{-8}^{-2} g(w)\,dw$. But then the dotted region would have the same area as the striped region. The dotted region is bigger. -so the actual ave. val. of g is <u>less than $g(-5)$</u>.

Common Error: if you don't draw a picture this is harder to see.

c) The key idea is $\frac{d}{dt} A_f(t) = g(t)$.

i) where g is pos., A is inc. $(-8,-2)$, $(3,5)$

ii) where g is inc., A is concave up $(-8,-6)$, $(-4,-3)$ $(2,4)$

iii) where g is neg & decr., A is decr. & concave down. $(-2,2)$

iv) $A_f(-2) = \int_0^{-2} g(w)\,dw = -\int_{-2}^{0} g(w)\,dw$

$\Rightarrow -[\text{negative}] = \text{positive}$

v) $\frac{d}{dt}\int_0^t g(w)\,dw = g(t)$

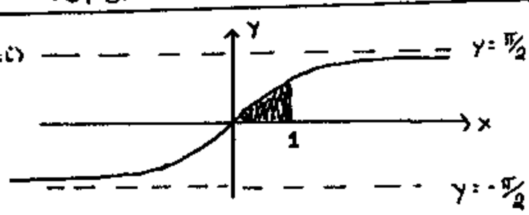
Details: $\frac{d}{dt}\int_0^t g(w)\,dw = \frac{d}{dt}\left[G(t) - G(0)\right]$

where $G' = g$.

$\frac{d}{dt}\left[G(t) - G(0)\right] = \frac{d}{dt}G(t) - \frac{d}{dt}[\text{const}] = g(t)$

Common Mistake: $A_f(t)$ is a function of t so it's derivative is a function of t, not of w!

#2. a i)



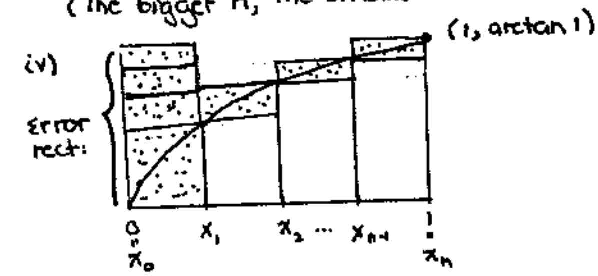
ii)

$L_n = \sum_{i=0}^{n-1} \arctan(x_i)\Delta x$; $R_n = \sum_{i=1}^{n} \arctan(x_i)\Delta x$

iii) $\int_0^1 \arctan x\,dx > L_n$, $\int_0^1 \arctan x\,dx < R_n$ } see picture

$L_n < L_{n+1}$, $R_n > R_{n+1}$ } since $\arctan x$ is increasing (its derivative, $\frac{1}{1+x^2}$, is positive) Left hd sums are lower bounds. But the larger the # of subdivisions, the better the left hd sum approximation is; so $L_n < L_{n+1}$. On the other hand, right hd sums are too big. But the accuracy is increased as the # of subdivisions increases, so $R_n > R_{n+1}$.
(The bigger n, the smaller the error.)

iv)



$$|R_n - L_n| = |f(b) - f(a)|\,\Delta x = |f(b)-f(a)|\,\frac{b-a}{n} = |\arctan 1 - \arctan 0|\,\frac{1}{n} = \left|\frac{\pi}{4} - 0\right|\cdot\frac{1}{n} = \frac{\pi}{4}\cdot\frac{1}{n}$$

We want $\frac{\pi}{4}\cdot\frac{1}{n} \le 0.01$, so let's solve $\frac{\pi}{4n} = .01$

$\frac{\pi}{4} = \frac{1}{100}n \Rightarrow n = \frac{\pi}{4}\cdot 100 \Rightarrow n = 25\pi$

But n <u>must</u> be an integer. $25\pi \approx 78.54 < 79$

Choose n to be the # 79.

Common Errors: some people said $n = 25\pi$. Do you really mean to partition $[0,1]$ into 25π equal pieces Some people wrote $\sum_{i=1}^{25\pi}$. What does that mean The biggest problem was "forgetting the error form But you can reconstruct this for yourselves! Either draw a picture (see above) - the area of the error rect. is $\left[\arctan 1 \cdot \frac{1}{n}\right]$, or subtract L_n from R_n to get $(f(b) - f(a))\Delta x$ -since all other terms cancel.

v) $L_{79} \approx .4338$; $R_{79} \approx .4437$

2.vi) If you average L_n and R_n you get the trapezoidal sum. This is an underestimate, since $\arctan x$ is concave down on $[0,1]$.

$f = \arctan x$
$f' = \frac{1}{1+x^2}$
$f'' = \frac{-2x}{(1+x^2)^2} \le 0$ for $x \in [0,1]$

b) (iii) $\frac{d}{dx}\left[x\arctan x - \frac{1}{2}\ln(1+x^2)\right] =$

$x \cdot \frac{1}{1+x^2} + 1 \cdot \arctan x - \frac{1}{2}\,\frac{1}{1+x^2}\cdot 2x =$

$\frac{x}{1+x^2} + \arctan x - \frac{x}{1+x^2} = \arctan x$

Common Error: Ignoring the Chain Rule

$\frac{d}{dx}\frac{[\arctan x]^2}{2} = \arctan x \cdot \frac{1}{1+x^2}$

$\frac{d}{dx}\left[\ln(1+x^2)\right] = \frac{1}{1+x^2}\cdot 2x$

$\frac{d}{dx}\left[\frac{1}{1+x^2}\right] = \frac{d}{dx}(1+x^2)^{-1} = -\frac{1}{(1+x^2)^2}\cdot 2x$

c) $\int_0^1 \arctan x\,dx = \left[x\arctan x - \frac{1}{2}\ln(1+x^2)\right]\Big|_0^1$

$= \arctan 1 - \frac{1}{2}\ln 2 - [0 - \ln(1)] = \frac{\pi}{4} - \frac{1}{2}\ln 2$

d) Numerically approximate your answer to (c) and make sure it falls between the bounds you established in (a v). Make sure your ans. to a(iii) & a(v) are consistent.

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* 3. a) $\int x e^{x^2+1}\,dx$ let $u = x^2+1$

$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\int e^u \frac{1}{2}du = \frac{1}{2}e^u + C$

$= \frac{1}{2}e^{x^2+1} + C$

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b) $\int \tan 5x\,dx = \int \frac{\sin(5x)}{\cos(5x)}\,dx$

let $u = \cos 5x$

$du = -5\sin 5x\,dx$

$-\frac{1}{5}du = \sin 5x\,dx$

$\int \frac{\sin 5x}{\cos 5x}\,dx = -\frac{1}{5}\int\frac{du}{u} = -\frac{1}{5}\ln|u| + C$

$= -\frac{1}{5}\ln|\cos(5x)| + C$

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c) $\int\left(\frac{3x+5}{x^2} + \frac{x}{x+2}\right)dx = \int\frac{3x+5}{x^2}\,dx + \int\frac{x}{x+2}\,dx$

We'll do each separately:

$\int\frac{3x+5}{x^2}\,dx = \int\left(\frac{3x}{x^2} + \frac{5}{x^2}\right)dx = \int\left(\frac{3}{x} + 5x^{-2}\right)dx$

$= 3\ln|x| + \frac{5x^{-1}}{-1} + C_1$

(Strategy ~ split up integral into pieces)

$\int\frac{x}{x+2}\,dx$: Strategy - use substitution to split up integral into pieces.

$u = x+2 \Rightarrow x = u-2$
$du = dx$

$\int\frac{x}{x+2}\,dx = \int\frac{u-2}{u}\,du = \int\left(\frac{u}{u} - \frac{2}{u}\right)du$

$= \int\left(1 - \frac{2}{u}\right)du = u - 2\ln|u| + C_2 = x+2 - 2\ln|x+2| + C$

final answer: $3\ln|x| - \frac{5}{x} + x + 2 - 2\ln|x+2| + C$ (2 can be omitted.)

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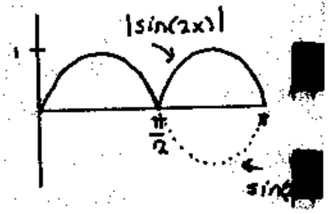
d) $\int_0^{\pi}|\sin(2x)|\,dx$



$= 2\int_0^{\pi/2}\sin(2x)\,dx$

$= 2\left[\frac{1}{2}(-\cos 2x)\Big|_0^{\pi/2}\right]$

$= -\cos 2x\Big|_0^{\pi/2} = -\cos(\pi) - (-\cos 0)$

$= -(-1) + 1 = 1+1 = 2$

Note: An ans. of 0 doesn't make sense!

Common Error: When dealing with $\int_a^b |f(x)|\,dx$ it is necessary to split the integral into pieces on which the sign of f doesn't change.

Here you could have $\int_0^{\pi/2}\sin(2x)\,dx + \int_{\pi/2}^{\pi}(-\sin 2x)$

Before integrating you've got to take off the absolute values - splitting the integrals into pieces where the integrand is pos. & where the integrand is n

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e) $\int_e^7 \frac{dx}{x\ln x}$ let $u = \ln x$, $du = \frac{1}{x}dx$ $\begin{cases}\text{when } x = 7,\ u = \ln 7\\ \text{when } x = e,\ u = \ln e = \end{cases}$

$\int_e^7\frac{dx}{x\ln x} = \int_1^{\ln 7}\frac{1}{u}\,du = \ln|u|\Big|_1^{\ln 7} = \ln|\ln 7| - \ln|$

$= \ln|\ln 7|$

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f) $\int_{-1/2}^{1/2}\frac{1}{1+4x^2}\,dx = \int_{-1/2}^{1/2}\frac{1}{1+(2x)^2}\,dx$

let $u = 2x$ $\begin{cases}\text{when } x = -\frac{1}{2},\ u = 2(-\frac{1}{2}) = -1\\ \text{when } x = \frac{1}{2},\ u = 2(\frac{1}{2}) = 1\end{cases}$

$du = 2dx$

$\frac{1}{2}du = dx$

$\int_{-1}^{1}\frac{1}{1+u^2}\frac{1}{2}du = \frac{1}{2}\int_{-1}^{1}\frac{1}{1+u^2}\,du = \frac{1}{2}\arctan(u)\Big|_{-1}^{1}$

$= \frac{1}{2}[\arctan 1 - \arctan(-1)] = \frac{1}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = \frac{2}{2}\,\frac{\pi}{4}$

$= \frac{\pi}{4}$.

Common Errors: Incorrect symmetry arguments were leaving folks with answers of 0. But if you integrate something that's always positive you can't get an answer of 0.

We know $\sin Bx$ and $\cos Bx$ are periodic with period $2\pi/B$, so $f(x)$ is $\boxed{\text{periodic}}$ with period $\boxed{2\pi/3}$.

$f'(x) = 3\cos 3x - 3\sin 3x$

$0 = 3(\cos 3x - \sin 3x)$

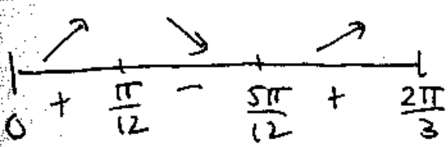
$\Longrightarrow \cos 3x = \sin 3x$

We know sin and cos are equal when their inputs are $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$

Thus, we have $3x = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$

So, $x = \frac{\pi}{12}, \frac{5\pi}{12}, \ldots$

Common mistake: $\sin 3x =$ something does not mean $\sin x =$ same thing!!



Since f is periodic, we only need to look at one period

$f(\pi/12) = \sin\frac{3\pi}{12} + \cos\frac{3\pi}{12} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

$f(5\pi/12) = \sin\frac{15\pi}{12} + \cos\frac{15\pi}{12} = \frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}} = -\sqrt{2}$

So, first abs max is $\boxed{\left(\frac{\pi}{12}, \sqrt{2}\right)}$.

And first abs min is $\boxed{\left(\frac{5\pi}{12}, -\sqrt{2}\right)}$

c) [No question. A typo.]

d) Find pts. of intersection:

$\sin 3x + \frac{1}{2} = \sin 3x + \cos 3x$

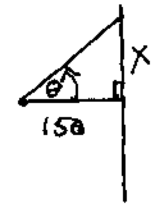
$\frac{1}{2} = \cos 3x$

$\Rightarrow 3x = \frac{\pi}{3}, \frac{5\pi}{3}, \ldots$

$x = \frac{\pi}{9}, \frac{5\pi}{9}$

In the shaded region, $g \geq f$, so area is $\boxed{\int_{\pi/9}^{5\pi/9} [g(x) - f(x)]\,dx}$.

5.



Know: $\frac{dx}{dt}$

Want: $\frac{d\theta}{dt}$

So, relate x, θ.

$\tan\theta = \frac{x}{150}$

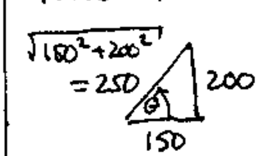
$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{x}{150}$

$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{150}\cdot\frac{dx}{dt}$

$\frac{d\theta}{dt} = \cos^2\theta \cdot \frac{1}{150}\cdot\frac{dx}{dt}$

At the moment in question, $\frac{dx}{dt} = -300$.

Now find $\cos\theta$:



$\cos\theta = \frac{150}{250} = \frac{3}{5}$

Notice it's a 3-4-5 triangle!

$\frac{d\theta}{dt} = \left(\frac{3}{5}\right)^2 \cdot \frac{1}{150}\cdot(-300) = \frac{-18}{25}$

We don't need the neg. sign because question just asks how fast camera turns, not how fast relative to something.

Units of θ are radians. Units of t are seconds. So, units of $\frac{d\theta}{dt}$ are $\frac{\text{rad}}{\text{sec}}$.

Answer: $\boxed{\frac{18}{25}\,\frac{\text{rad}}{\text{sec}}}$

Common mistake: Many people tried to find rate of camera turning as the deriv of a distance, but since camera is turning, we need an angle instead.

6.

a) Amt at $t=6$ is

orig amt + net change

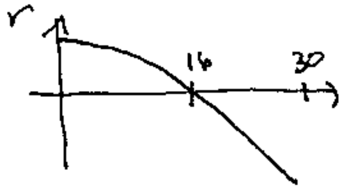
$$= 900 + \int_0^6 \left[8 - \frac{t^2}{32}\right] dt$$

$$= 900 + \left[8t - \frac{t^3}{96}\right]_0^6$$

$$= 900 + \left[\left(48 - \frac{216}{96}\right) - (0 - 0)\right]$$

$$= \boxed{\$945.75 \text{ million}}$$

b) When is it max? Look at $r(t)$.



So, amt is incr until $t=16$ and decr afterwards. Max at $t=16$.

$$\text{Max amt} = 900 + \int_0^{16} r(t)\,dt$$

$$= 900 + \left[8t - \frac{t^3}{96}\right]_0^{16}$$

$$= 900 + \left[\left(128 - \frac{4096}{96}\right) - (0 - 0)\right]$$

$$= \boxed{\$985\tfrac{1}{3} \text{ million}}$$

c) From picture above, min is at $t=0$ or $t=30$. At $t=0$, there are \$900 million. At $t=30$:

$$900 + \int_0^{30} r(t)\,dt$$

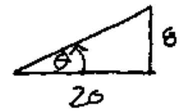
$$= 900 + \left[8t - \frac{t^3}{96}\right]_0^{30}$$

$$= 900 + \left[\left(240 - \frac{27000}{96}\right) - (0 - 0)\right]$$

$$= \boxed{\$858.75 \text{ million}}$$ Choose this b/c it's < 900.

7.

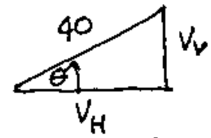
a)



$$\tan\theta = \frac{8}{20} = \frac{2}{5}$$

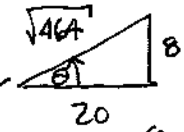
$$\Rightarrow \boxed{\theta = \arctan\frac{2}{5}}$$

b)



$$\sin\theta = \frac{V_V}{40} \Rightarrow V_V = 40\sin\theta$$
$$\cos\theta = \frac{V_H}{40} \Rightarrow V_H = 40\cos\theta$$

Compute $\sin\theta$, $\cos\theta$ from drawing in (a), shown below



$$\sin\theta = \frac{8}{\sqrt{464}}$$
$$\cos\theta = \frac{20}{\sqrt{464}}$$

$$V_V = 40 \cdot \frac{8}{\sqrt{464}} = \boxed{\frac{320}{\sqrt{464}}}$$

$$V_H = 40 \cdot \frac{20}{\sqrt{464}} = \boxed{\frac{800}{\sqrt{464}}}$$

c) distance = rate · time

$$\Rightarrow t = \frac{20}{\frac{800}{\sqrt{464}}} = \boxed{\frac{\sqrt{464}}{40} \approx .538 \text{ sec}}$$

d) Plug in $t = \frac{\sqrt{464}}{40}$ from (c).

$$h\left(\frac{\sqrt{464}}{40}\right) = -16 \cdot \frac{464}{1600} + \overbrace{\frac{320}{\sqrt{464}}}^{V_V} \cdot \frac{\sqrt{464}}{40} + 5$$

$$= -4.64 + 8 + 5$$

$$= \boxed{8.36 \text{ ft}}$$

<u>Common mistakes</u>: Many people gave decimal approximations in (b) rather than simplifying their expressions to get exact values.