1. (15 points) Below is a graph of the function g.
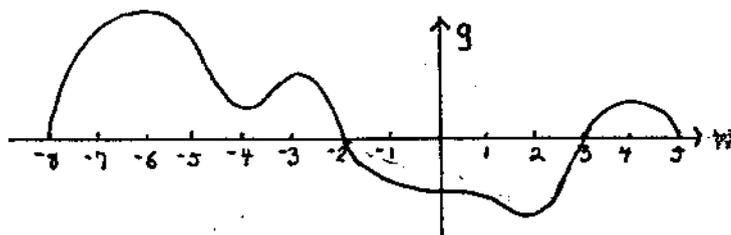
a) Put the following in ascending order, with "<" or "=" signs between the letters corresponding to the integrals.

$$A = \int_{-8}^{-2} g(w)dw \qquad B = \int_{-2}^{-8} g(w)dw \qquad C = \int_{-3}^{-2} g(w)dw \qquad D = \int_{-3}^{3} g(w)dw \qquad E = \int_{-3}^{4} g(w)dw$$

Answer : _____

b) (i) Write an expression that gives the average value of g on the interval [-8, -2]. _____

   (ii) Is the average value of g on the interval [-8, -2] greater than g(-5), less than g(-5) or equal to g(-5)?

c) Let $A_f(t) = \int_0^t g(w)\,dw$.

   i) Where on [-8, 5] is $A_f(t)$ increasing? _____

   ii) Where on [-8, 5] is $A_f(t)$ concave up? _____

   iii) Where on [-8, 5] is $A_f(t)$ concave down and decreasing? _____

   iv) Is $A_f(-2)$ positive, negative, or zero? Explain your reasoning.

   v) Find $\dfrac{d}{dt}\int_0^t g(w)\,dw$.

2. (23 points) Please note: Successful completion of parts (b) and (c) of this problem does NOT depend upon successful completion of part (a).

a) i) Sketch the graph of $y = \arctan x$. Please label any vertical or horizontal asymptotes. Shade in the area corresponding to the integral $\int_0^1 \arctan x \, dx$.

ii) Partition the interval $[0,1]$ into n equal subintervals, each of length Δx.
Let $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x$, ..., $x_k = k\,\Delta x$, ..., $x_n = n\Delta x = 1$.
Use summation notation to write L_n and R_n, the left and right hand Riemann Sums approximating $\int_0^1 \arctan x \, dx$.

$L_n = $ _____ $R_n = $ _____

iii) In the blanks write $<$, $>$ or $=$, whichever makes the statement true for all positive integers n where $0 < n < \infty$. If your choice of $<$, $>$ or $=$ depends on the value of n, then write "?" in the blank.
Note: L_n and R_n refer to the sums from part (ii).

$\int_0^1 \arctan x \, dx$ _____ L_n

$\int_0^1 \arctan x \, dx$ _____ R_n

L_n _____ L_{n+1}

R_n _____ R_{n+1}

iv) What is the smallest value of n that will give left and right hand sums that differ from one another by no more than 0.01? Show your work in finding this n.

v) Using left and right hand sums and the value of n you found in (iv), compute upper and lower bounds for $\int_0^1 \arctan x \, dx$ that differ from one another by no more than 0.01. You need not show your work.

vi) If you take the average of the upper and lower bounds you found in part (v), will you have an overestimate of $\int_0^1 \arctan x \, dx$, or an underestimate? What criterion are you using to decide?

b) Which one of the following is an antiderivative of arctan x?

Explain clearly how you know your answer is correct.

i) $\dfrac{(\arctan x)^2}{2}$ ii) $\ln(1 + x^2)$ iii) $x \arctan x - \dfrac{1}{2} \ln(1 + x^2)$ iv) $\dfrac{1}{1 + x^2}$

c) Evaluate $\displaystyle\int_0^1 \arctan x\ dx$. Give an exact answer that is simplified as much as possible. In this part of the problem we are *not* interested in a numerical approximation.

d) Look back over this problem. Are your answers consistent? If they are not consistent, which ones do you think are incorrect? (If you can, change them to something correct!)

3. (18 points) Integrate.

a) $\displaystyle\int x\, e^{x^2+1}\ dx$

b) $\displaystyle\int \tan(5x)\ dx$

c) $\displaystyle\int \left(\dfrac{3x + 5}{x^2} + \dfrac{x}{x + 2} \right) dx$

d) $\displaystyle\int_0^{\pi} |\sin(2x)|\ dx$

e) $\displaystyle\int_e^7 \dfrac{dx}{x \ln x}$

f) $\displaystyle\int_{-0.5}^{0.5} \dfrac{1}{1 + 4x^2}\ dx$

4. (13 points) Let $f(x) = \sin 3x + \cos 3x$ on $x \geq 0$.

a) Is f periodic? If so, what is the fundamental period of f?

b) Find the absolute maximum and minimum **values** (y-values) of f and the smallest x-value corresponding to each.

Fill in the blanks below.

The absolute **maximum** value of f is _____. (Give an exact answer, not a numerical approximation.)
The smallest x-value for which this maximum value is attained is_____ . (Give an exact answer.)

The absolute **minimum** value of f is _____. (Give an exact answer, not a numerical approximation.)
The smallest x-value for which this minimum value is attained is _____ . (Give an exact answer.)

d) Write an integral (or sum / difference of integrals) that give(s) the area shaded below. The area is bounded above by $g(x) = \sin(3x) + .5$ and below by $f(x) = \sin 3x + \cos 3x$. Do NOT evaluate the integral; just set it up. (Include exact endpoints of integration!)
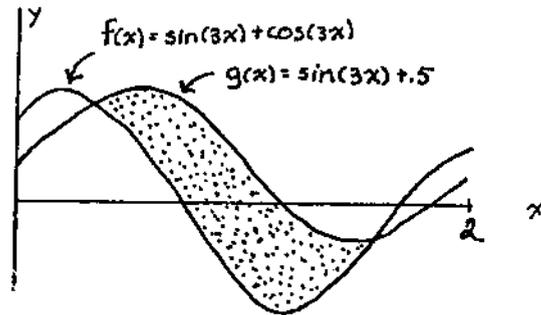
5. (11 points) You are seated 150 feet from the finish line of an auto drag strip, as shown below. You have a video camera focused on a particular car and you rotate the camera to keep the car in the center of the lens. When the car is 200 feet from the finish line and travelling at 300 feet/second, at what rate is the camera turning? Include units in your answer.
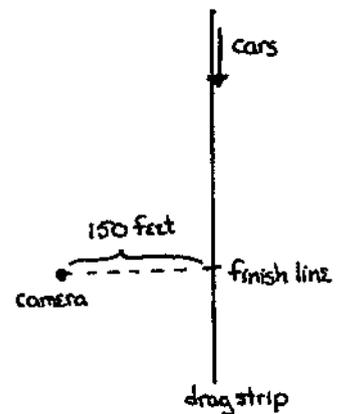
6. (11 points) The rate at which money flows into a mutual fund is modelled by $r(t) = 8 - \frac{t^2}{32}$ millions of dollars per day on $0 \le t \le 30$. When r is positive it means there is a net inflow, and when r is negative there is a net outflow. At $t = 0$ there are already 900 million dollars in the fund.

a) How much is in the fund at $t = 6$?

b) What is the maximum amount in the fund on $[0, 30]$?

c) What is the minimum amount in the fund on $[0, 30]$?

7. (10 points) It is a steamy hot day in Cambridge and a young boy decides that it is the perfect time to take out his water pistol. He stands at his open window and points his water pistol (which is five feet above ground level) at his friend's bedroom window. The friend's window is 8 feet above the level of the water pistol and is across an alley 20 feet wide. See the picture given below.

a) At what angle is the boy pointing his water pistol? We are looking for the angle θ shown.

b) When he shoots the pistol, water comes out with an initial velocity of 40 feet/sec. (Recall that we can represent this by a vector of length 40 making an angle of θ with the horizontal.)
What is the horizontal component of the initial velocity? (We'll refer to this number as v_h.)
What is the vertical component of the initial velocity? (We'll refer to this number as v_v.)

v_h = _____feet/sec v_v = _____feet/sec

c) The water hits the wall of his friend's house when it has travelled a horizontal distance of 20 feet. The horizontal component of the water's velocity stays constant at v_h ft/sec. Use your answer to (b) to find the number of seconds it takes the water to reach his friend's house. Give an exact answer and then a numerical approximation. For the latter, give two decimal places.

d) When the boy points his water pistol at his friend's window he is not thinking about the effects of gravity on the water. Instead of going into his friend's bedroom window (13 feet above the ground) the water hits his friend's house at a lower height. Using the work we have done in the course we know that the height of the leading end of the water at time t is given by

$$h(t) = -16t^2 + \text{(the initial vertical component of velocity)} \, t + 5,$$

where t is the number of seconds since the trigger was pulled. Find the height at which the water hits the wall. Give an exact answer and then a numerical approximation. For the latter, give two decimal places.