Annotated Solutions to the

First Examination: March 9, 1998

1. Amplitude $= 4 = |A|$
Balance value $= 7 = D$ $\Big\}$ $y = A\cos Bt + D$

Use cosine flipped vertically

Period $= 4 = \frac{2\pi}{B} \Rightarrow B = \frac{2\pi}{4} = \frac{\pi}{2}$

$y = -4\cos\left(\frac{\pi}{2}t\right) + 7$

Common Error: calculating the period incorrectly. One and $\frac{1}{4}$ cycles have been completed in 5 units: $\frac{5}{4}$ periods $= 5$ units

$1$ period $= 4$ units

2. $f(x) = \pi x\,(2x+5)^{-3x}$

$\ln f(x) = \ln\pi + \ln x - 3x\ln(2x+5)$

$\frac{1}{f(x)}f'(x) = \frac{1}{x} - 3\ln(2x+5) - 3x\cdot\frac{2}{2x+5}$

$f'(x) = f(x)\left[\frac{1}{x} - 3\ln(2x+5) - \frac{6x}{2x+5}\right]$

$f'(x) = \pi x(2x+5)^{-3x}\left[\frac{1}{x} - 3\ln(2x+5) - \frac{6x}{2x+5}\right]$

$f'(2) = 2\pi\cdot 9^{-6}\left[\frac{1}{2} - 3\ln 9 - \frac{12}{9}\right]$

$f'(2) = \frac{2\pi}{9^6}\left[-\frac{5}{6} - 3\ln 9\right]$

3. a. $\sum_{k=2}^{\infty}\frac{99}{3^k} = \frac{99}{9} + \frac{99}{27} + \frac{99}{3^4} + \cdots$

i) First two non-zero terms: $11 + \frac{11}{3}$

ii) This is a geometric series with $r = \frac{1}{3}$

It converges (since $|r| = \frac{1}{3} < 1$) to

$\frac{a}{1-r} = \frac{11}{1-\frac{1}{3}} = \frac{11}{\frac{2}{3}} = \frac{33}{2} = 16.5$

b. $\sum_{k=1}^{\infty}\left(1-\frac{2}{k}\right)^k = (1-2) + (1-1)^2 + \left(1-\frac{2}{3}\right)^3 + \left(1-\frac{2}{4}\right)^4 + \cdots$

i) First two non-zero terms: $-1 + \frac{1}{27}$

ii) $\lim_{k\to\infty}\left(1-\frac{2}{k}\right)^k = e^{-2}$

iii) This series diverges since the terms of the series approach $e^{-2}$ as $k\to\infty$, and $e^{-2}\neq 0$. Since the terms don't approach zero we know the series diverges. It (the sum) grows without bound.

Common Errors

(1) $\lim_{k\to\infty}\left(1-\frac{2}{k}\right)^k = e^{-2}$ Not $e^2$

(2) Although the terms of the series tend $\to e^{-2}$, the series, which is the SUM of the terms, diverges. Many people confuse the series with the terms of the series.

(3) Some people said that since the series is not geometric it will diverge. But there are non-geometric series, like $\sum_{n=1}\frac{1}{n2^n}$, that converge.

3c. $\sum_{k=0}^{\infty}\frac{3^k}{2^{k+2}} = \sum_{k=0}^{\infty}\frac{1}{4}\left(\frac{3}{2}\right)^k = \frac{1}{4} + \frac{3}{8} + \frac{9}{16}\cdots$

i) First two non-zero terms: $\frac{1}{4} + \frac{3}{8}$

ii) Diverges. This is a geometric series w/ $|r| = \frac{3}{2} > 1$ so it diverges.

d) $\sum_{k=0}^{\infty}\sin\left(k\frac{\pi}{2}\right) = \sin 0 + \sin\frac{\pi}{2} + \sin\pi + \sin\frac{3\pi}{2} + \sin\frac{4\pi}{2} + \cdots$

$= 0 + 1 + 0 - 1 + 0 + 1 + \cdots$

$= 1 - 1 + 1 - 1 + 1 - \cdots$

The nonzero terms of the series oscillate between 1 and $-1$.

i) First two non-zero terms $1 - 1$

ii) The series diverges - the partial sums oscillate between +1 and 0. (The terms don't $\to 0$ as $k\to\infty$.)

Common Errors Some people ignored the sine! Others confused the terms of the series with the partial sums.

For clarification - consider the harmonic series

$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$

The terms $\to 0$ but the partial sums $\longrightarrow\infty$

in (b) the terms $\to e^{-2}$ but the partial sums $\to\infty$

in (d) the terms oscillate between 1 & -1; the partial sums oscillate between 1 and 0.

4. $\tan 3x = 1$ for $0\le x\le 2\pi$

let $u = 3x$ $\quad 0\le 3x\le 6\pi$

$0\le u\le 6\pi$


$\tan u = 1$

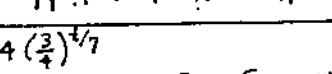
$u = \begin{cases}\frac{\pi}{4},\ \frac{5\pi}{4}\\ \frac{\pi}{4}+2\pi,\ \frac{5\pi}{4}+2\pi\\ \frac{\pi}{4}+4\pi,\ \frac{5\pi}{4}+4\pi\end{cases}$ or $u = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}$

$u = 3x$ so $x = \underbrace{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}}_{\text{six answers in all.}}$

Common (Very common) Error:

People got $\frac{\pi}{4}$ & $\frac{5\pi}{4}$ for $u$ but then got only $\frac{\pi}{12}$ & $\frac{5\pi}{12}$ for $x$. If $x\in[0,2\pi]$ then $u = 3x\in[0,6\pi]$.

look at




6 answers

7. $M(t) = 4\left(\frac{3}{4}\right)^{t/7}$

Right after taking a pill: Geom. Series w/ $a = 4$, $r = (3/4)^{1/7}$

$4 + 4(3/4)^{1/7} + 4(3/4)^{2/7} + \cdots = \frac{a}{1-r} = \frac{4}{1-(3/4)^{1/7}}$

$\approx 99.343$ mg

Right before a pill $\approx 99.343 - 4 = 95.343$ mg.

Common Error $\frac{1}{4}$ is eliminated, so 3/4 is left.

$\frac{3}{4}$ is left after 7 days, not 1 day

pill just taken

5.a. 60000 is the PV of the 15 payments.
Let $m$ = amt. of each payment.

$$60000 = \frac{m}{(1+\frac{.06}{12})^{12\cdot 3}} + \cdots + \frac{m}{(1+\frac{.06}{12})^{12\cdot 17}}$$

$$\frac{1}{1.005^{12}} 60000 = \frac{m}{1.005^{12\cdot 4}} + \cdots + \frac{m}{1.005^{12\cdot 18}}$$

$$\left(1-\frac{1}{1.005^{12}}\right) 60000 = m\left(1.005^{-12\cdot 3} - 1.005^{-12\cdot 18}\right)$$

$$m = \frac{60000(1-1.005^{-12})}{1.005^{-12\cdot 3} - 1.005^{-12\cdot 18}} \approx \boxed{\$7039.87}$$

Common errors: Many said $r = 1.005^{12}$ instead of $1.005^{-12}$.
Many forgot the 12's in the exponents.
Many said last payment is $t=18$ instead of 17.

b. Same thing except it's infinite series.

$$60000 = \frac{m}{1.005^{12\cdot 3}} + \frac{m}{1.005^{12\cdot 4}} + \cdots$$

$r = \frac{1}{1.005^{12}}$, so it converges.

$$\Rightarrow 60000 = \frac{a}{1-r} = \frac{m\cdot 1.005^{-12\cdot 3}}{1-1.005^{-12}}$$

$$\Rightarrow m = 60000 \cdot 1.005^{12\cdot 3} \cdot (1-1.005^{-12})$$

$$\approx \boxed{\$4171.24}$$

6.a. First simplify: $\ln(xy^2) = \ln x + 2\ln y$

$$\frac{d}{dx}[\ln x + 2\ln y] = \frac{d}{dx}[-xy+2]$$

$$\frac{1}{x} + 2\cdot\frac{1}{y}\cdot\frac{dy}{dx} = -1\cdot y - x\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{2}{y}+x\right) = -y - \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{-y - 1/x}{2/y + x}$$

Plug in $(4, 1/2)$ to get $\frac{dy}{dx} = -\frac{3}{32}$

Now find $b$: $y = mx + b$

$$\frac{1}{2} = -\frac{3}{32}\cdot 4 + b \Rightarrow b = 7/8$$

$$\boxed{y = -3/32\, x + 7/8}$$

Common error: forgetting to find $b$!

6.b. slope $= \frac{dy}{dx} = \frac{-y - 1/x}{2/y + x} = 0$.

Thus, $-y - \frac{1}{x} = 0 \Rightarrow y = -\frac{1}{x}$.

Plug this back into original eq'n to find the pts. on the curve where y'

$$\ln(xy^2) = -xy + 2$$

$$\ln\left(x\left(-\tfrac{1}{x}\right)^2\right) = -x\cdot\frac{-1}{x} + 2$$

$$\ln\frac{1}{x} = 3$$

$$-\ln x = 3 \Rightarrow \ln x = -3 \Rightarrow x =$$

And since $y = -\frac{1}{x}$, we get $y = \frac{-1}{e^{3}}$

$$\Rightarrow \boxed{\left(\frac{1}{e^3}, -e^3\right)}$$

Check that doesn't make undef'd: C

Common errors: $\frac{1}{\frac{2}{y}+x} \neq \frac{y}{2} + \frac{1}{x}$

Also, many people didn't realize that a fraction is 0 if the numerator is 0 (and denom $\neq 0$).

8. a. Since amt is decreasing, we know $\frac{dQ}{dt}$ is neg. Since $\frac{dQ}{dt} = \lambda\underbrace{(Q-20)}_{pos}$, we see $\lambda$ is neg. $Q-20$ is decr, so $\underbrace{\lambda(Q-20)}_{=\frac{dQ}{dt}}$ is incr $\Rightarrow$ $\boxed{\text{concave}}$

b. $-30 = \lambda(260-20) \Rightarrow \lambda = -1/8$

$$\frac{dQ}{dt} = -\frac{1}{8}(100-20) = \boxed{-10 \text{ moles/hr}}$$

c. Let $y = Q - 20$. So, $\frac{dQ}{dt} = \lambda(Q-20)$ becomes $\frac{dy}{dt} = \lambda y$. We know the soln is $y = Ce^{\lambda t} = Ce^{-t/8}$.

$$\Rightarrow Q - 20 = Ce^{-t/8} \Rightarrow Q = 20 + Ce$$

Find C: $Q(0) = 260$

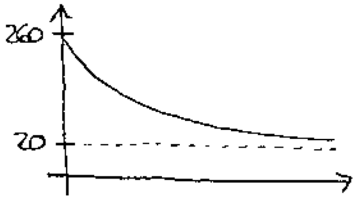
$$260 = 20 + Ce^{0}$$

$$\Rightarrow C = 240$$

$$\boxed{Q(t) = 20 + 240e^{-t/8}}$$

(cont'd)

8.d.




As $t \to \infty$, $\boxed{Q \to 20.}$

Common error: Saying in (a) that Q is conc. down, the drawing the opposite in (d)! (Or vice-versa.)

9.a


Know: $\frac{dx}{dt}$
Want: $\frac{dz}{dt}$

$$\frac{d}{dt}\left[z^2\right] = \frac{d}{dt}\left[90^2 + x^2\right]$$

$$\cancel{2}z\frac{dz}{dt} = 0 + \cancel{2}x\frac{dx}{dt}$$

Now plug in:

$x = 60$, $z = \sqrt{60^2+90^2} = 10\sqrt{117}$,

$\frac{dx}{dt} = -25$

$$10\sqrt{117}\,\frac{dz}{dt} = 60(-25)$$

$$\boxed{\frac{dz}{dt} = \frac{-150}{\sqrt{117}} \approx -13.87}$$

b.


Want: $\frac{dz}{dt}$
Know: $\frac{dx}{dt}$, $\frac{dy}{dt}$

$$\frac{d}{dt}z^2 = \frac{d}{dt}\left[x^2+y^2\right]$$

$$\cancel{2}z\frac{dz}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$$

Plug in: $x=45$ $\frac{dx}{dt} = -30$ Note the neg. sign!!

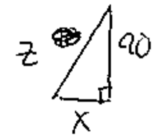
$y=30$ $\frac{dy}{dt} = 25$

$$z = \sqrt{30^2+45^2} = 5\sqrt{117}$$

$$\Rightarrow \boxed{\frac{dz}{dt} = \frac{-120}{\sqrt{117}} \approx -11.09}$$

Very common error: writing $\frac{dx}{dt} = +30$ instead of $-30$.

c.




$$\frac{90}{z} = \sin 60^\circ = \frac{\sqrt{3}}{2} \Rightarrow z = \boxed{\frac{180}{\sqrt{3}}}$$

$$\frac{90}{x} = \tan 60^\circ = \sqrt{3} \Rightarrow x = \boxed{\frac{90}{\sqrt{3}}}$$

10. a) $\frac{dV}{dt} = kV\cdot(800-V)$ where k is a positive constant

b) at $V=200$ $\frac{dV}{dt} = k\cdot 200\cdot 600 = 120{,}000k$

at $V=300$ $\frac{dV}{dt} = k\cdot 300\cdot 500 = 150{,}000k$

The rate at which the virus is spreading is greater when $V=300$ than when $V=200$.

$$\%\text{ change} = \frac{\text{final} - \text{initial}}{\text{initial}} = \frac{\text{change}}{\text{initial}} = \frac{30{,}000k}{120{,}000k}$$

$$= \frac{3}{12} = \frac{1}{4} = .25 = 25\%$$

So there has been a 25% increase.

Common Errors

% change doesn't mean express A as a % of B -
% change - divide by initial not final value
$.2 = 20\%$ Not $2\%$.

c) $\frac{dV}{dt} = 800kV - kV^2$

$$\frac{d^2V}{dt^2} = 800k\frac{dV}{dt} - 2kV\frac{dV}{dt}$$

$$\frac{d^2V}{dt^2} = (800k - 2kV)\underbrace{kV(800-V)}_{\frac{dV}{dt}}$$

Common Error $\frac{d}{dt}800kV = 800k\frac{dV}{dt}$ Not $800k$

$\frac{d}{dt}V^2 = 2V\frac{dV}{dt}$.