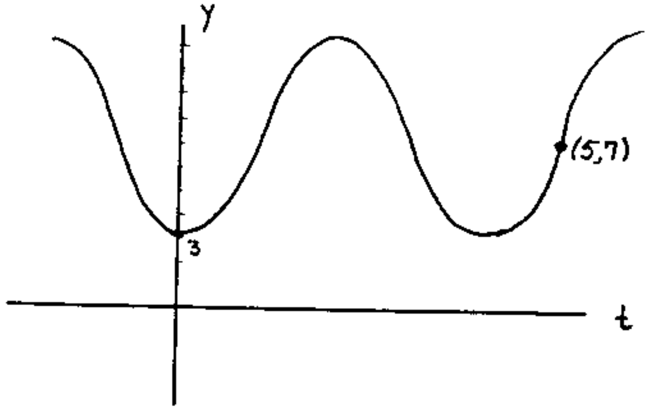
First Examination
Mathematics Xb
March 9, 1998

1. (7 points) Write a possible equation for the trigonometric graph given below.



2. (8 points) Find $f'(2)$ if $f(x) = \pi\, x(2x+5)^{-3x}$. Give an **exact** answer, not a numerical approximation.

3. (9 points) For each of the following infinite series, first write out (and simplify) the first two non-zero terms of the series and then determine whether the series converges or diverges. In either case, explain your reasoning clearly and completely. If the series converges, tell us to what it converges. If the series diverges, tell us why (and how) it diverges.

a) $\sum_{k=2}^{\infty} \frac{99}{3^n}$

i) First two non-zero terms: ________ ________

ii) Convergence / Divergence of series:

b) $\sum_{k=1}^{\infty} \left(1 - \frac{2}{k}\right)^k$

i) First two non-zero terms: ________ ________

ii) Evaluate $\lim_{k \to \infty} \left(1 - \frac{2}{k}\right)^k$.

iii) Convergence / Divergence of series:

c) $\sum_{k=0}^{\infty} \frac{3^k}{2^{k+2}}$

i) First two non-zero terms: ________ ________

ii) Convergence / Divergence of series:

d) $\sum_{k=0}^{\infty} \sin(k\pi/2)$

i) First two non-zero terms: ________ ________

ii) Convergence / Divergence of series:

4. (7 points) Find all x between 0 and 2π such that $\tan(3x) = 1$. Please give exact answers. Show your work.

5. (12 points) You've won an award of $60,000. Today you put it in a bank account with a guaranteed nominal interest rate of 6% compounded monthly. You plan to withdraw a fixed amount of money from the account once a year beginning exactly three years from today.

a) How much can you withdraw each year if you plan to deplete the account after the 15th withdrawal? If your answer involves a sum, please put it in closed form. Then give your answer to the nearest penny.

b) Suppose a fixed amount of money will be withdrawn every year beginning exactly three years from today but continuing forever. What is the maximum amount that can be withdrawn each year? Give your answer to the nearest penny. Be sure to give an answer that ensures that the account is never depleted.

6. (10 points)

a) Find the equation of the tangent line to $\ln(xy^2) = -xy + 2$ at the point (4, 1/2).

b) At what point(s) does the graph of $\ln(xy^2) = -xy + 2$ have a horizontal tangent line? Please give the x- and y-coordinates of the point(s).

7. (9 points) A medication is eliminated from the body at a rate proportional to the amount in the bloodstream. 1/4 of the amount of medication in the bloodstream is eliminated every 7 days. If a person takes 4 mg of the medication every day indefinitely, then in the long run what is the amount of the medication in his bloodstream immediately after taking a pill? Immediately before taking a pill?

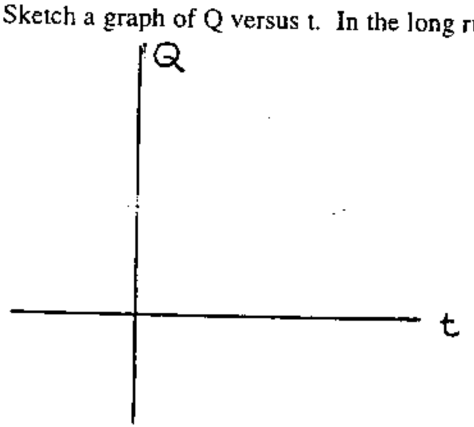
8. (13 points) Suppose that $\frac{dQ}{dt} = \lambda(Q - 20)$ where Q is the quantity of a certain substance measured in moles and t is time measured in hours. When t = 0 we know that Q is 260 moles and that the number of moles is decreasing with time.

a) Will the graph of Q(t) be linear, concave up, or concave down? Explain your reasoning.

b) Suppose that Q is decreasing at a rate of 30 moles per hour when t = 0. At what rate is Q decreasing when Q is 100?

c) Find the particular solution to the differential equation $\frac{dQ}{dt} = \lambda(Q - 20)$ corresponding to the information given in the statement of the problem and in part (b). There should be no arbitrary constants in your answer.

d) Sketch a graph of Q versus t. In the long run, what happens to Q?

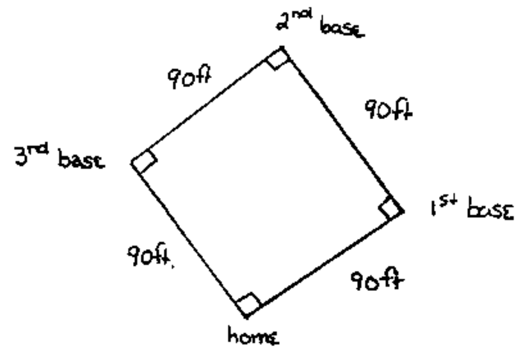


9. (14 points)

A baseball diamond is a square whose sides are 90 feet long. The batter stands at home plate to hit the ball and runs from home to first base.

John Valentin is up at bat. He hits the ball and takes off for first base.

a) At the moment when he has run a third of the distance between home and first base he is moving at 25 ft/sec. At this moment, at what rate is his distance from second base decreasing?

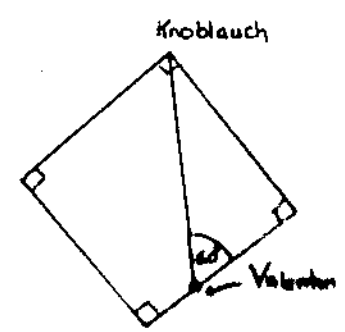


b) There is a man on third base who is running towards home. At the instant that Valentin has run a third of the distance between home and first base, our man on third is halfway home and is running at 30 ft/sec. At this instant, how fast is the distance between Valentin and this man changing? Is the distance increasing or decreasing?

c) Chuck Knoblauch is standing on second base and watching Valentin as he runs towards first. When the angle between his line of sight and that of Valentin's line of motion is 60 degrees (see picture below), exactly how far apart are the two men and exactly how far is Valentin from first base? Please give exact answers, not numerical approximations.

Distance between Knoblauch and Valentin :________________

Distance between Valentin and first base: ________________



10. (11 points) The rate at which a virus spreads among a company's 800 computers is proportional to the product of the number of computers infected and the number uninfected. Let V = V(t) be the number of computers infected at time t.

a) Write a differential equation reflecting the situation described.

b) Is the rate at which the virus is spreading greater when 200 computers are infected, or when 300 computers are infected? Explain. By what percent has the rate at which computers are being infected changed from V=200 to V = 300?

c) Using your answer to (a), find $\frac{d^2V}{dt^2}$. (Hint: Don't forget that V varies with t!)