Assignment #9

2/26/2001

$\boxed{18.5}$

#3 a) Future Value = (Present V) e^{kt}

$$6000 = M_0 e^{0.04t} \qquad t=5$$

$$M_0 = \frac{6000}{e^{0.04(5)}} = \$4912.38$$

b) Sum the present values of the payments

$$S = \underbrace{\frac{2000}{e^{0.04(3)}}}_{1^{st}\text{ payment}} + \frac{2000}{e^{0.04(4)}} + \underbrace{\frac{2000}{e^{0.04(5)}}}_{3^{rd}\text{ payment}}$$

$$= \boxed{\$5115.59}$$

#7 (see handout solutions on last page)

$$M_0 = P = \$1866.28$$

#14 the sum of the present values of the 8 \$10,000 withdrawals should equal the amount they must put in today.

$$\$10,000 = (PV)\left(1+\frac{r}{n}\right)^{nt} = (PV)\left(1+\frac{0.06}{4}\right)^{4t}$$

$$\underbrace{PV}_{\text{each withdrawal}} = \frac{\$10,000}{(1.015)^{4t}}$$

$$S = \frac{\$10,000}{(1.015)^{4(9)}} + \frac{10,000}{(1.015)^{4(10)}} + \dots + \frac{10,000}{(1.015)^{4(16)}}$$

geometric series with $r = \frac{1}{1.015^4}$

$$S = \frac{\frac{10{,}000}{(1.015)^{36}} - \frac{10{,}000}{(1.015)^{68}}}{1 - \frac{1}{1.015^4}} = \boxed{\$38{,}355.13}$$

#24 Level of Drug is the sum of what is leftover from each of the 28 pills.

After 28^{th} pill ...

How much left of 28^{th}? $\Rightarrow M$

" " 27^{th}? $\Rightarrow M\left(\frac{1}{2}\right)^{\frac{12}{24}}$

" " 26^{th}? $\Rightarrow M\left(\frac{1}{2}\right)^{\frac{24}{24}}$

⋮

1^{st}? $\Rightarrow M\left(\frac{1}{2}\right)^{\frac{27 \cdot 12}{24}}$

$$S = M + M\left(\tfrac{1}{2}\right)^{\frac{1}{2}} + M\left(\tfrac{1}{2}\right)^{1} + \dots M\left(\tfrac{1}{2}\right)^{\frac{27 \cdot 12}{24}}$$

$$a = M \;;\; r = \left(\tfrac{1}{2}\right)^{1/2} = \sqrt{\tfrac{1}{2}}$$

$$S = \frac{M - M\left(\frac{1}{2}\right)^{\frac{28 \cdot 12}{24}}}{1 - \sqrt{\frac{1}{2}}} = \boxed{\frac{M\left(1 - \left(\frac{1}{2}\right)^{14}\right)}{1 - \frac{\sqrt{2}}{2}}}$$ exact answer

approximate answer: $\boxed{S = 3.4\,M}$

#25 "taken indefinitely" means "infinite series"

$M=$ milligrams in each pill.

If amount of drug in body is maximum, it must be IMMEDIATELY after taking a pill.

$$\text{Amount of Drug (max)} = L = \underset{\substack{\uparrow \\ \text{most recent} \\ \text{pill}}}{M} + \underset{\substack{\uparrow \\ \text{pill one} \\ \text{day ago}}}{M\left(\tfrac{1}{2}\right)^{\frac{1}{10}}} + \underset{\substack{\uparrow \\ \text{pill 2} \\ \text{days ago}}}{M\left(\tfrac{1}{2}\right)^{\frac{2}{10}}} \ldots\ldots$$

$$\text{Max amount of Drug} = L = \frac{a}{1-r}$$

$$L = \frac{M}{1-\left(\frac{1}{2}\right)^{\frac{1}{10}}}$$

$$\#\text{of milligrams per pill} = M = L\left[1-\left(\tfrac{1}{2}\right)^{\frac{1}{10}}\right]$$

$$= \boxed{0.067\ L}$$

Problem Set #9

At this point we expect you to have read all but 18.3 from Chapter 18. When you do the reading you should be armed with a pencil and paper so that you can work examples before reading through the solutions.

This problem set is meant to focus your attention on "setting up" problems. Think about what is going on. Make time lines for yourself. Or construct tables. You need to establish a time frame for the problem and must be able to explain the rationale behind your approach. In order to show that we really mean this, no credit will be given on either homework or exams for taking stabs at a problem by just writing down the "formula" for the sum of a geometric series. Before writing a geometric sum in closed form you must clearly write out the sum you are working with. And you must make some illuminating commentary concerning the meaning of the sum. That's the problem-solving task!

The assignment is Section18.5 #3, 7 (see note below), 14, 24, 25.

As part of #7, please answer the questions before the table, fill in the table below and then answer the questions that follow. When filling in the table, using the distributive law and avoiding nested parentheses will help you see the pattern. Once you see the pattern you can use "$+\cdots+$". Let P be the amount of the payments. The problem is asking you to find P. We want you to look at this from many angles.

Preliminary Question:

Give upper and lower bounds for the amount that you will have to pay. Explain your reasoning briefly.

lower = 0 upper = Future Value of \$10,000 = $(10{,}000)(1.07)^{14}$.

time (in years)	amount you owe (from year 5 on, make this the amount you owe immediately after a payment)
0	\$10,000
1	$(\$10{,}000)(1.07)$
2	$(\$10{,}000)(1.07) = (10{,}000)(1.07)^2$
3	$(\$10{,}000)(1.07)^3$
4	$(\$10{,}000)(1.07)^4$
5	$(\$10{,}000)(1.07)^5 - P$
6	$(\$10{,}000)(1.07)^6 - P(1.07) - P$
7	$(\$10{,}000)(1.07)^7 - P(1.07)^2 - P(1.07) - P$
8	$(\$10{,}000)(1.07)^8 - P(1.07)^3 - P(1.07)^2 - P(1.07) - P$
9	$(\$10{,}000)(1.07)^9 - P(1.07)^4 - P(1.07)^3 - \ldots - P$
10	$(\$10{,}000)(1.07)^{10} - P(1.07)^5 - \ldots - P$
11	$(\$10{,}000)(1.07)^{11} - P(1.07)^6 - \ldots - P$
12	$(\$10{,}000)(1.07)^{12} - P(1.07)^7 - \ldots - P$
13	$(\$10{,}000)(1.07)^{13} - P(1.07)^8 - \ldots - P$
14	$(\$10{,}000)(1.07)^{14} - P(1.07)^9 - \ldots P = 0$ *

(over) so, $(\$10{,}000)(1.07)^{14} = P + P(1.07) + P(1.07)^2 + \ldots + P(1.07)^9$

Questions:

(a) Why does the table stop at 14? because @ $t=14$, the last payment is made

(b) What is the meaning (in words) of the last line of the table? The amount you owe $= 0$.
What is the last time of the table equal to? 0

(answer below) (c) Use your answer to the last question to solve for P.
When you use a loan or mortgage calculator on a computer you are provided with a table like the one below. Fill it in, using the answer you got for P.

yearly payment	\$1866.28
total amount paid out	10(1866.28) = \$18,662.80
total amount of interest paid	18662.80 − 10,000 = \$866.80

(d) Sketch the loan and payments on a time line.
Find the present value of each of the payments.
Sum the present values of each of the ten payments. What do you get? Does this make sense? Show how this is mathematically equivalent to what you did with the last line of the table.
Notice that this gives you an alternative way of doing the problem.

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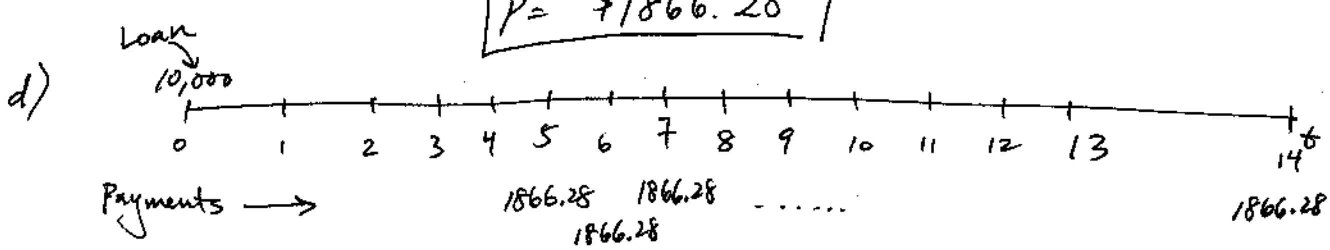
c) $$(\$10{,}000)(1.07)^{14} = P + \underbrace{P(1.07) + P(1.07)^2 + P(1.07)^3 + \dots + P(1.07)^9}_{\text{geometric } \cancel{\text{sum}} \text{ series } (a=P,\ r=1.07)}$$

$$(\$10{,}000)(1.07)^{14} = \frac{P - P(1.07)^{10}}{1-1.07}$$

$$(1-1.07)(10{,}000)(1.07)^{14} = P(1-1.07^{10})$$

$$\boxed{P = \$1866.28}$$

d)



$$PV \text{ of payments} = \frac{1866.28}{(1.07)^5} + \frac{1866.28}{(1.07)^6} + \dots + \frac{1866.28}{(1.07)^{14}}$$

$$S = \frac{\frac{1866.28}{(1.07)^5} - \frac{1866.28}{(1.07)^{15}}}{1 - \frac{1}{1.07}} \qquad r = \frac{1}{1.07}$$

$$= \boxed{\$10{,}000}$$ ← This makes sense because you SHOULD completely pay back the amount you borrowed TODAY.