Homework Assignment 11: Solutions

1. Figures 1 and 2 from the homework assignment show the locations (i.e. the x-coordinates) of all critical and inflection points of the function $f(x)$. Recall from Math Xa that:

- The critical points of the original function $f(x)$ are located at the points where the derivative $f'(x)$ is equal to zero, and,
- The inflection points of the original function $f(x)$ are located at the points where the second derivative $f''(x)$ changes sign. (Remember that it is not enough for the second derivative just to be equal to zero – for the original function to have an inflection point, the second derivative must actually change signs.)

The x-values of the points where either $f'(x)$ is equal to zero or where $f''(x)$ changes sign can be read from Figures 1 and 2 on the homework and are listed in Table 1 (below).

x-coordinate	Type of point
-1	Local minimum
$\frac{-1}{\sqrt{3}} \approx -0.577$	Inflection point
0	Local maximum
$\frac{1}{\sqrt{3}} \approx 0.577$	Inflection point
1	Local minimum

Table 1.

The nature of each of the critical points (maximum or minimum) can be inferred from either Figure 1 or Figure 2. If you wanted to use Figure 1 to classify the critical points, then you would pay attention to the sign of the derivative on the left and the right of the critical point:

Sign of derivative on left Of critical point	Sign of derivative on right Of critical point	Nature of critical point
+	−	Local maximum
−	+	Local minimum

If you wanted to use Figure 2 to classify the critical points, you would pay attention to the sign of the second derivative at the x-coordinates where the critical points are located:

Sign of second derivative at x-coordinate Of critical point	Nature of critical point
−	Local maximum
+	Local minimum

Putting the information contained in Table 1 together with the given fact that $f(0) = 1$ yields the graph shown in Figure 1. Note that the y-coordinates of the points (with the exception of the point (0, 1)) are not specified because it is very hard (at this point) to know exactly what the precise y-coordinates of the points actually are.

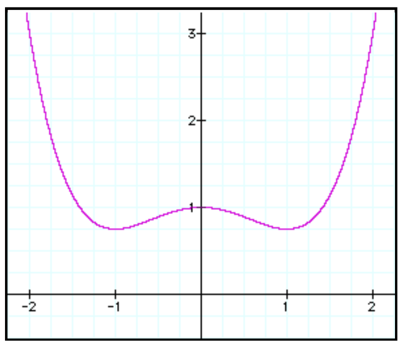

Figure 1: Qualitative sketch of graph of original function based on first and second derivative graphs.

2. Based on the locations of the x-intercepts of $y = f'(x)$ from Figure 1 in the homework assignment (they are located at $x = -1$, $x = 0$ and $x = 1$) and the shape of the curve near each of the x-intercepts (the graph cuts straight through suggesting that each x-intercept has multiplicity 1) give the equation:

$$f'(x) = k \cdot (x+1) \cdot (x-0) \cdot (x-1).$$

In the homework assignment, the value of the constant of proportionality (k) was given to be $k = 1$. The equation for the first derivative is therefore:

$$f'(x) = (x+1) \cdot (x-0) \cdot (x-1) = x^3 - x.$$

Calculating this antiderivative of $f'(x)$ will give the original function, $f(x)$. the antiderivative of the polynomial written out above is:

$$f(x) = \tfrac{1}{4} \cdot x^4 - \tfrac{1}{2} \cdot x^2 + C$$

where C is a constant. To evaluate the constant C you can use the given fact that $f(0) = 1$. Substituting $x = 0$ and $f(0) = 1$ into the equation for $f(x)$ gives:

$$1 = f(0) = \tfrac{1}{4} \cdot (0)^4 - \tfrac{1}{2} \cdot (0)^2 + C = C.$$

So, $C = 1$ and the equation for the original function $f(x)$ is:

$$f(x) = \tfrac{1}{4} \cdot x^4 - \tfrac{1}{2} \cdot x^2 + 1.$$

3. The graph of the function $y = f(x) = \tfrac{1}{4} \cdot x^4 - \tfrac{1}{2} \cdot x^2 + 1$ is shown in Figure 2 below.

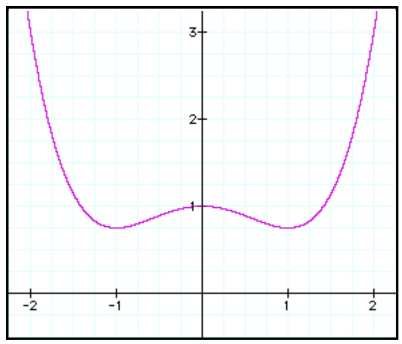

Figure 2: Precise graph of $y = f(x)$.

A comparison between the graphs shown in Figure 1 and Figure 2 reveals that all of the critical and inflection points occur in identical locations (at least as far as the x-coordinates are concerned). Likewise, the nature of each critical point (local maximum, local minimum) is the same in Figure 1 and Figure 2. The only discrepancies are the y-coordinates of the points, and this is only because a qualitative sketch of the graph of the original function (as you made in Question 1) is not overly concerned with the precise height of each point on the sketch of the graph of $y = f(x)$.

4. Based on the appearance of Figure 6 in the homework assignment, any polynomial with a degree of greater than 2 should do a reasonable job of representing the patterns in the data. The equations the you should obtain from using a quadratic, cubic or quartic equation are given in Table 2 (below) and all of these graphs are shown (together with the data points) in Figure 3 (below).

In this problem, I will use T to represent the time (in hours) since the test subject ingested the glucose powder, and $g(T)$ to represent the blood glucose (in units of mg per dl).

Polynomial used	Equation
Quadratic	$g(T) = -63.42857 \cdot T^2 + 118.45714 \cdot T + 79.28571$
Cubic	$g(T) = 9.3333 \cdot T^3 - 91.425857 \cdot T^2 + 138.5238 \cdot T + 77.885714$
Quartic	$g(T) = 98.6666 \cdot T^4 - 385.3333 \cdot T^3 + 391.3333 \cdot T^2 - 37.6666 \cdot T + 80$

Table 2: Possible polynomial functions for subject who had ingested glucose.

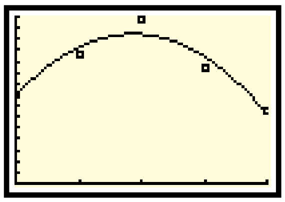

Figure 3(a): Scatter plot of data points and graph of quadratic function.

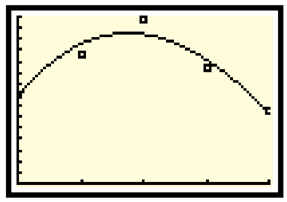

Figure 3(b): Scatter plot of data points and graph of cubic function.

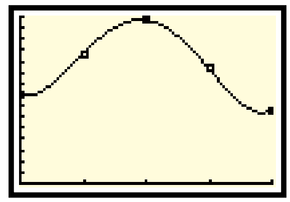

Figure 3(c): Scatter plot of data points and graph of quartic function.

Of the functions given in Table 2, the quartic function gives the best correlation coefficient ($R^2 = 1$) and resembles the original graph (Figure 6 in the homework assignment) reasonably closely. (You are free to use any of these polynomial functions, however, as they all do a pretty good job of representing the patterns in the blood glucose level.)

The antiderivatives of the three polynomial functions defined in Table 2 (above) are shown in Table 3 (below). In each case, the antiderivative has been named[1] $G(T)$, and C is intended to represent a constant whose value is not specified.

Polynomial used	Antiderivative of polynomial
Quadratic	$G(T) = \frac{-63.42857}{3} \cdot T^3 + \frac{118.45714}{2} \cdot T^2 + 79.28571 \cdot T + C$
Cubic	$G(T) = \frac{9.3333}{4} \cdot T^4 - \frac{91.425857}{3} \cdot T^3 + \frac{138.5238}{2} \cdot T^2 + 77.885714 \cdot T + C$
Quartic	$G(T) = \frac{98.6666}{5} \cdot T^5 - \frac{385.3333}{4} \cdot T^4 + \frac{391.3333}{3} \cdot T^3 - \frac{37.6666}{2} \cdot T^2 + 80 \cdot T + C$

Table 3: Antiderivatives of polynomial functions from Table 2.

The *glycemic index* is equal to the area under the graph of $y = g(T)$ between $T = 0$ and $T = 2$. Using the Fundamental Theorem of Calculus, this area will be equal to

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[1] The convention of writing the function in lower case (e.g. $g(T)$) and the antiderivative of the function in upper case (i.e. $G(T)$) is one followed by many authors.

the change in the value of the function $G(T)$ between $T = 0$ and $T = 2$. In terms of integral notation, this could be expressed as:

$$Glycemic\ index = \int_0^2 g(T) \cdot dT = G(2) - G(0).$$

The values of the glycemic index that can be obtained using each of the functions listed in Table 3 are given in Table 4 below.

Polynomial used	Glycemic index, $G(2) - G(0)$
Quadratic	226.3428
Cubic	226.3499
Quartic	218.3553

Table 4: Glycemic indicex for 50g of glucose.

5. Based on the appearance of Figure 7 in the homework assignment, any polynomial with a degree of greater than 2 should do a reasonable job of representing the patterns in the data. The equations the you should obtain from using a quadratic, cubic or quartic equation are given in Table 5 (below) and all of these graphs are shown (together with the data points) in Figure 4 (below).

In this problem, I will use T to represent the time (in hours) since the test subject ingested the two slices of white bread, and $b(T)$ to represent the blood glucose (in units of mg per dl).

Polynomial used	Equation
Quadratic	$b(T) = -45.7143 \cdot T^2 + 99.02857 \cdot T + 78.94286$
Cubic	$b(T) = 2.6666 \cdot T^3 - 53.71428 \cdot T^2 + 104.7619 \cdot T + 78.54286$
Quartic	$b(T) = 68 \cdot T^4 - 269.3333 \cdot T^3 + 279 \cdot T^2 - 16.6666 \cdot T + 80$

Table 5: Possible polynomial functions for subject who had ingested white bread.

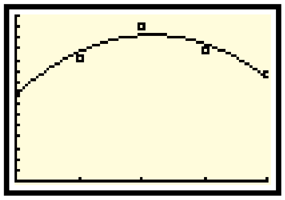

Figure 4(a): Scatter plot of data points and graph of quadratic function.

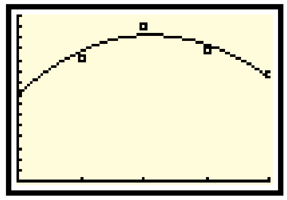

Figure 4(b): Scatter plot of data points and graph of cubic function.

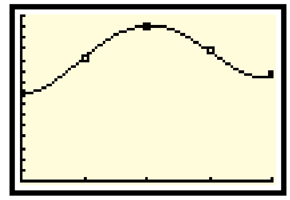

Figure 4(c): Scatter plot of data points and graph of quartic function.

Of the functions given in Table 5, the quartic function gives the best correlation coefficient ($R^2 = 1$) and resembles the original graph (Figure 7 in the homework assignment) reasonably closely. (You are free to use any of these polynomial functions, however, as they all do a pretty good job of representing the patterns in the blood glucose level.)

The antiderivatives of the three polynomial functions defined in Table 5 (above) are shown in Table 6 (below). In each case, the antiderivative has been named $B(T)$, and C is intended to represent a constant whose value is not specified.

Polynomial used	Antiderivative of polynomial
Quadratic	$B(T) = \frac{-45.7143}{3} \cdot T^3 + \frac{99.02857}{2} \cdot T^2 + 78.94286 \cdot T + C$
Cubic	$B(T) = \frac{2.6666}{4} \cdot T^4 - \frac{53.71428}{3} \cdot T^3 + \frac{104.7619}{2} \cdot T^2 + 78.54286 \cdot T + C$
Quartic	$B(T) = \frac{68}{5} \cdot T^5 - \frac{269.3333}{4} \cdot T^4 + \frac{279}{3} \cdot T^3 - \frac{16.6666}{2} \cdot T^2 + 80 \cdot T + C$

Table 6: Antiderivatives of polynomial functions from Table 5.

The *glycemic index* is equal to the area under the graph of $y = b(T)$ between $T = 0$ and $T = 2$. Using the Fundamental Theorem of Calculus, this area will be equal to the change in the value of the function $B(T)$ between $T = 0$ and $T = 2$. In terms of integral notation, this could be expressed as:

$$\text{Glycemic index} = \int_0^2 b(T) \cdot dT = B(2) - B(0).$$

The values of the glycemic index that can be obtained using each of the functions listed in Table 6 are given in Table 7 below.

Polynomial used	Glycemic index, $B(2) - B(0)$
Quadratic	234.03806
Cubic	234.03544
Quartic	228.5336

Table 7: Glycemic index for two slices of white bread.