Homework Assignment 14: Solutions

The technique of u-substitution is the algorithm for calculating antiderivatives described below.

- Identify the "inside function."
- Set the inside function equal to u.
- Calculate the derivative $\frac{du}{dx}$.
- Re-arrange the derivative to make dx the subject of the expression.
- Substitute u for the "inside function" and the expression that you have just created for dx into the indefinite integral.
- You should now have an indefinite integral expressed entirely in terms of u with no x's left in it.
- Find the antiderivatve.
- Substitute the "inside function" for u in the antiderivative.

In Problems 1-5 we will use **bold face** to indicate where we are listing:

- An explicit statement of the "inside function."
- An explicit calculation of the derivative $\frac{du}{dx}$.
- Steps that can be used to re-arrange the derivative with the objective of making dx the subject.
- The indefinite integral expressed entirely in terms of u.
- The final answer expressed entirely in terms of x.

1. In this problem, you were asked to find a formula for the antiderivative :

$$\int 4\cdot\left[9\cdot e^{x}+\ln(x)\right]^{3}\cdot\left(9\cdot e^{x}+\tfrac{1}{x}\right)\cdot dx$$

When the expression within the integral notation is conveniently written out in the format given here, **identifying a good choice of the "inside function"** is reasonably straight-forward. The function:

$$u=9\cdot e^{x}+\ln(x)$$

sits inside another algebraic expression, suggesting that it might make a good choice for the "inside function." Another clue suggesting that this choice is a good one is the fact that the derivative of this function,

$$\left(9\cdot e^{x}+\tfrac{1}{x}\right),$$

appears as a factor in the indefinite integral.

The next step in performing the technique of u-substitution is to **calculate the derivative**, $\frac{du}{dx}$. Differentiating the "inside function" gives:

$$\frac{du}{dx} = 9 \cdot e^x + \frac{1}{x}.$$

Next, the idea is to pretend that the derivative $\frac{du}{dx}$ is like a fraction and **re-arrange the derivative to make *dx* the subject of the equation.** To do this, you can multiply both sides of the derivative equation by *dx*:

$$du = \left(9 \cdot e^x + \tfrac{1}{x}\right) \cdot dx,$$

and then divide both sides of the equation by the quantity in the brackets to get:

$$dx = \frac{du}{\left(9 \cdot e^x + \frac{1}{x}\right)}.$$

Now that we have identified the "inside function" and found an expression for *dx*, the next step in performing the technique of u-substitution is to **substitute into the indefinite integral so that the indefinite integral is expressed entirely in terms of *u* with no *x*'s left in it.** We will do this in two steps in the hope of improving clarity, but there is no reason why you shouldn't do this in one step. The first step will be to replace:

$$9 \cdot e^x + \ln(x)$$

by *u* in the indefinite integral:

$$\int 4 \cdot \left[9 \cdot e^x + \ln(x)\right]^3 \cdot \left(9 \cdot e^x + \tfrac{1}{x}\right) \cdot dx = \int 4 \cdot u^3 \cdot \left(9 \cdot e^x + \tfrac{1}{x}\right) \cdot dx.$$

Next, we will replace *dx* in this indefinite integral with $\dfrac{du}{\left(9 \cdot e^x + \frac{1}{x}\right)}$:

$$\int 4 \cdot u^3 \cdot \left(9 \cdot e^x + \tfrac{1}{x}\right) \cdot dx = \int 4 \cdot u^3 \cdot \left(9 \cdot e^x + \tfrac{1}{x}\right) \cdot \frac{du}{\left(9 \cdot e^x + \frac{1}{x}\right)}.$$

Simplifying this indefinite integral by canceling the quantities $\left(9 \cdot e^x + \frac{1}{x}\right)$ gives:

$$\int 4 \cdot u^3 \cdot \left(9 \cdot e^x + \tfrac{1}{x}\right) \cdot \frac{du}{\left(9 \cdot e^x + \frac{1}{x}\right)} = \int 4 \cdot u^3 \cdot du.$$

Now that the indefinite integral has been transformed into a (much less complicated) indefinite integral expressed entirely in terms of u, we will **find the antiderivative.**

$$\int 4 \cdot u^3 \cdot du = u^4 + C.$$

Lastly, to complete the problem, we **use the inside function to express the antiderivative in terms of *x*.** Since:

$$u = 9 \cdot e^x + \ln(x),$$

the antiderivative is therefore equal to:

$$u^4 + C = \left[9 \cdot e^x + \ln(x)\right]^4 + C.$$

To summarize what we have actually done in all of these steps: we started out with the indefinite integral:

$$\int 4 \cdot \left[9 \cdot e^x + \ln(x)\right]^3 \cdot \left(9 \cdot e^x + \tfrac{1}{x}\right) \cdot dx.$$

Our ultimate objective here is to find a formula for the antiderivative of:

$$f(x) = 4 \cdot \left[9 \cdot e^x + \ln(x)\right]^3 \cdot \left(9 \cdot e^x + \tfrac{1}{x}\right).$$

This is what the steps in the technique of u-substitution are designed to do – to find a formula for the antiderivative of this $f(x)$. If we have done all of the steps in the technique of u-substitution correctly, then the antiderivative $F(x)$ will be given by:

$$F(x) = \left[9 \cdot e^x + \ln(x)\right]^4 + C.$$

2. In this problem, you were asked to find a formula for the antiderivative :

$$\int \tfrac{1}{2} \cdot \left[x^{10} + \ln(x)\right]^{\frac{-1}{2}} \cdot \left(10 \cdot x^9 + \tfrac{1}{x}\right) \cdot dx.$$

When the expression within the integral notation is conveniently written out in the format given here, **identifying a good choice of the "inside function"** is reasonably straight-forward. The function:

$$u = x^{10} + \ln(x)$$

sits inside another algebraic expression, suggesting that it might make a good choice for the "inside function." Another clue suggesting that this choice is a good one is the fact that the derivative of this function,

$$\left(10\cdot x^9+\tfrac{1}{x}\right),$$

appears as a factor in the indefinite integral.

The next step in performing the technique of u-substitution is to **calculate the derivative**, $\frac{du}{dx}$. Differentiating the "inside function" gives:

$$\frac{du}{dx}=10\cdot x^9+\frac{1}{x}.$$

Next, the idea is to pretend that the derivative $\frac{du}{dx}$ is like a fraction and **re-arrange the derivative to make *dx* the subject of the equation.** To do this, you can multiply both sides of the derivative equation by *dx*:

$$du=\left(10\cdot x^9+\tfrac{1}{x}\right)\cdot dx,$$

and then divide both sides of the equation by the quantity in the brackets to get:

$$dx=\frac{du}{\left(10\cdot x^9+\tfrac{1}{x}\right)}.$$

Now that we have identified the "inside function" and found an expression for *dx*, the next step in performing the technique of u-substitution is to **substitute into the indefinite integral so that the indefinite integral is expressed entirely in terms of *u* with no *x*'s left in it.** We will do this in two steps in the hope of improving clarity, but there is no reason why you shouldn't do this in one step if you feel confident. The first step will be to replace

$$x^{10}+\ln(x)$$

by *u* in the indefinite integral:

$$\int \tfrac{1}{2}\cdot\left[x^{10}+\ln(x)\right]^{\frac{-1}{2}}\cdot\left(10\cdot x^9+\tfrac{1}{x}\right)\cdot dx=\int \tfrac{1}{2}\cdot u^{\frac{-1}{2}}\cdot\left(10\cdot x^9+\tfrac{1}{x}\right)\cdot dx.$$

Next, we will replace *dx* in this indefinite integral with $dx=\frac{du}{\left(10\cdot x^9+\tfrac{1}{x}\right)}$:

$$\int \tfrac{1}{2}\cdot u^{\frac{-1}{2}}\cdot\left(10\cdot x^9+\tfrac{1}{x}\right)\cdot dx=\int \tfrac{1}{2}\cdot u^{\frac{-1}{2}}\cdot\left(10\cdot x^9+\tfrac{1}{x}\right)\cdot\frac{du}{\left(10\cdot x^9+\tfrac{1}{x}\right)}.$$

Simplifying this indefinite integral by canceling the quantities $\left(10\cdot x^9+\frac{1}{x}\right)$ gives:

$$\int \tfrac{1}{2}\cdot u^{\frac{-1}{2}}\cdot\left(10\cdot x^9+\tfrac{1}{x}\right)\cdot\frac{du}{\left(10\cdot x^9+\frac{1}{x}\right)}=\int \tfrac{1}{2}\cdot u^{\frac{-1}{2}}\cdot du.$$

Now that the indefinite integral has been transformed into a (much less complicated) indefinite integral expressed entirely in terms of u, we will **find the antiderivative.**

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Lastly, to complete the problem, we **use the inside function to express the antiderivative in terms of *x*.** Since:

$$u=x^{10}+\ln(x),$$

the antiderivative is therefore equal to:

$$u^{\frac{1}{2}}+C=\left[x^{10}+\ln(x)\right]^{\frac{1}{2}}+C.$$

Again, let's look back on the steps that we have done and summarize the work: we started out with the indefinite integral:

$$\int \tfrac{1}{2}\cdot\left[x^{10}+\ln(x)\right]^{\frac{-1}{2}}\cdot\left(10\cdot x^9+\tfrac{1}{x}\right)\cdot dx.$$

Our ultimate objective here is to find a formula for the antiderivative of:

$$f(x)=\tfrac{1}{2}\cdot\left[x^{10}+\ln(x)\right]^{\frac{-1}{2}}\cdot\left(10\cdot x^9+\tfrac{1}{x}\right).$$

This is what the steps in the technique of u-substitution are designed to do – to find a formula for the antiderivative of this $f(x)$. If we have done all of the steps in the technique of u-substitution correctly, then the antiderivative $F(x)$ will be given by:

$$F(x)=\left[x^{10}+\ln(x)\right]^{\frac{1}{2}}+C.$$

3. In this problem, you were asked to find a formula for the antiderivative :

$$\int \frac{1}{\left[\frac{1}{x}+\frac{1}{x^2}\right]} \cdot \left(\frac{-1}{x^2}+\frac{-2}{x^3}\right) \cdot dx.$$

When the expression within the integral notation is conveniently written out in the format given here, **identifying a good choice of the "inside function"** is reasonably straight-forward. The function:

$$u = \frac{1}{x}+\frac{1}{x^2}$$

sits inside another algebraic expression, suggesting that it might make a good choice for the "inside function." Another clue suggesting that this choice is a good one is the fact that the derivative of this function,

$$\left(\frac{-1}{x^2}+\frac{-2}{x^3}\right),$$

appears as a factor in the indefinite integral.

The next step in performing the technique of u-substitution is to **calculate the derivative,** $\frac{du}{dx}$. Differentiating the "inside function" gives:

$$\frac{du}{dx} = \frac{-1}{x^2}+\frac{-2}{x^3}.$$

Next, the idea is to pretend that the derivative $\frac{du}{dx}$ is like a fraction and **re-arrange the derivative to make *dx* the subject of the equation.** To do this, you can multiply both sides of the derivative equation by *dx*:

$$du = \left(\frac{-1}{x^2}+\frac{-2}{x^3}\right) \cdot dx,$$

and then divide both sides of the equation by the quantity in the brackets to get:

$$dx = du \Big/ \left(\frac{-1}{x^2}+\frac{-2}{x^3}\right).$$

Now that we have identified the "inside function" and found an expression for *dx*, the next step in performing the technique of u-substitution is to **substitute into the indefinite integral so that the indefinite integral is expressed entirely in terms of *u* with no *x*'s left in it.** We will do this in two steps in the hope of improving clarity, but there is no reason why you shouldn't do this in one step if you feel confident. The first step will be to replace

$$\frac{1}{x}+\frac{1}{x^2}$$

by *u* in the indefinite integral gives:

$$\int \frac{1}{\left[\frac{1}{x}+\frac{1}{x^2}\right]} \cdot \left(\frac{-1}{x^2}+\frac{-2}{x^3}\right) \cdot dx = \int \frac{1}{u} \cdot \left(\frac{-1}{x^2}+\frac{-2}{x^3}\right) \cdot dx .$$

Next, we will replace dx in this indefinite integral with $dx = du \Big/ \left(\frac{-1}{x^2}+\frac{-2}{x^3}\right)$:

$$\int \frac{1}{u} \cdot \left(\frac{-1}{x^2}+\frac{-2}{x^3}\right) \cdot dx = \int \frac{1}{u} \cdot du,$$

where we have simplified the indefinite integral by canceling the quantities $\frac{1}{x}+\frac{1}{x^2}$. Now that the indefinite integral has been transformed into a (much less complicated) indefinite integral expressed entirely in terms of u, we will **find the antiderivative.**

$$\int \frac{1}{u} \cdot du = \ln(u) + C .$$

Lastly, to complete the problem, we **use the inside function to express the antiderivative in terms of *x*.** Since:

$$u = \frac{1}{x}+\frac{1}{x^2},$$

the antiderivative is therefore equal to:

$$\ln(u) + C = \ln\left(\frac{1}{x}+\frac{1}{x^2}\right) + C .$$

Again, let's look back on the steps that we have done and summarize the work: we started out with the indefinite integral:

$$\int \frac{1}{\left[\frac{1}{x}+\frac{1}{x^2}\right]} \cdot \left(\frac{-1}{x^2}+\frac{-2}{x^3}\right) \cdot dx .$$

Our ultimate objective here is to find a formula for the antiderivative of:

$$f(x) = \frac{1}{\left[\frac{1}{x}+\frac{1}{x^2}\right]} \cdot \left(\frac{-1}{x^2}+\frac{-2}{x^3}\right).$$

This is what the steps in the technique of u-substitution are designed to do – to find a formula for the antiderivative of this $f(x)$. If we have done all of the steps in the technique of u-substitution correctly, then the antiderivative $F(x)$ will be given by:

$$F(x) = \ln\left(\frac{1}{x}+\frac{1}{x^2}\right) + C .$$

4. In this problem, you were asked to find a formula for the antiderivative :

$$\int \frac{x}{\sqrt{1+x^2}} \cdot dx$$

In this problem (as in Question 5) it is quite a lot harder to get started because the indefinite integral has not been carefully laid out to "lay bare its secrets" and make a good choice for the "inside function" readily apparent. In this kind of a situation, usually the best that you can do is to try to **make a reasonable guess at what the "inside function" might be.** If you happen to make a good guess, then the problem should work out nicely. If you make a less than optimal choice for the "inside function" then it will be impossible to eliminate all of the x's from the indefinite integral. If that happens to you, then the best idea is to go back to square one and try a different choice for the "inside function." Everyone makes a sub-optimal guess at the "inside function" every now and again, although you will probably notice that you will get a *lot* better at making good guesses for "inside functions" as you complete more and more u-substitution problems.

In the case of the indefinite integral,

$$\int \frac{x}{\sqrt{1+x^2}} \cdot dx,$$

the only thing that I can see that is "inside" another algebraic formula is $1 + x^2$. I am going to **guess** that:

$$u = 1 + x^2$$

is the "inside function" and see how this guess works out.

The next step in performing the technique of u-substitution is to **find the derivative,** $\frac{du}{dx}$. Calculating this derivative gives:

$$\frac{du}{dx} = 2 \cdot x.$$

Next, I will **re-arrange the derivative to make *dx* the subject of the equation**.

$$dx = \frac{du}{2 \cdot x}.$$

Now I will try to do the crucial step that will determine whether my choice of $u = 1 + x^2$ as the "inside function" was a wise choice or not. I will attempt to **substitute *u* and the expression for *dx* into the indefinite integral with the intention of obtaining a less complicated indefinite integral that is expressed entirely in terms of *u*.** As before, I will do this in two steps for the sake of clarity, although if you feel confident in your abilities, there is no reason for you not to do this in one step. First, I will use $u = 1 + x^2$ to replace the $1 + x^2$ in the indefinite integral by u:

$$\int \frac{x}{\sqrt{1+x^2}} \cdot dx = \int \frac{x}{\sqrt{u}} \cdot dx.$$

Second, I will use the formula $dx = \dfrac{du}{2 \cdot x}$ to replace the dx in the indefinite integral:

$$\int \frac{x}{\sqrt{u}} \cdot dx = \int \frac{x}{\sqrt{u}} \cdot \frac{du}{2 \cdot x}.$$

Simplifying this indefinite integral by canceling the x's and carefully positioning the constant factor of $\frac{1}{2}$ within the indefinite integral gives:

$$\int \frac{x}{\sqrt{u}} \cdot \frac{du}{2 \cdot x} = \int \frac{1}{\sqrt{u}} \cdot \frac{du}{2} = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{2} \cdot du = \int \frac{1}{2} \cdot \frac{1}{\sqrt{u}} \cdot du = \int \tfrac{1}{2} \cdot u^{\frac{-1}{2}} \cdot du.$$

As it has been possible to cancel all of the x's out of the indefinite integral, leaving only a less complicated indefinite integral that is expressed entirely in terms of u, the guess of:

$$u = 1 + x^2$$

for the "inside function" was a pretty good guess. **Working out this less complicated indefinite integral** gives:

$$\int \tfrac{1}{2} \cdot u^{\frac{-1}{2}} \cdot du = u^{\frac{1}{2}} + C.$$

To complete the problem, all that remains is to **use the inside function to express this antiderivative entirely in terms of *x*.**

$$u^{\frac{1}{2}} + C = \left(1 + x^2\right)^{\frac{1}{2}} + C.$$

Therefore, the antiderivative of the function:

$$f(x) = \frac{x}{\sqrt{1+x^2}},$$

is given by:

$$F(x) = \left(1 + x^2\right)^{\frac{1}{2}} + C.$$

5. In this problem, you were asked to find a formula for the antiderivative :

$$\int x \cdot e^{x^2} \cdot dx$$

As in Question 4, the way that this indefinite integral has been written down is not particularly suggestive of an "inside function." This indefinite integral is perhaps even more difficult than the one in Question 4 as there is no obvious "inside" anywhere to be seen.

There is an "inside" here, but it is not that readily apparent. Re-writing the indefinite integral with one addition set of (strictly speaking, unnecessary) brackets can help to reveal where the "inside" is:

$$\int x \cdot e^{\left[x^2\right]} \cdot dx.$$

Therefore, the "inside" is actually located in the exponent of the exponential function. Based on this, a **reasonable guess for the "inside function"** might be:

$$u = x^2.$$

Working with this guess, the next step in performing the technique of u-substitution is to **calculate the derivative,** $\frac{du}{dx}$. Calculating this derivative gives:

$$\frac{du}{dx} = 2 \cdot x.$$

Next, I will **re-arrange the derivative to make *dx* the subject of the equation**.

$$dx = \frac{du}{2 \cdot x}.$$

Now comes the crucial step where we will see whether $u = x^2$ was a wise choice for the "inside function" or not. I will attempt to **substitute *u* and the expression for *dx* into the indefinite integral with the intention of obtaining a less complicated indefinite integral that is expressed entirely in terms of *u*.** As before, I will do this in two steps for the sake of clarity, although if you feel confident in your abilities, there is no reason for you not to do this in one step. First, I will use $u = x^2$ to replace the x^2 in the indefinite integral by u:

$$\int x \cdot e^{\left[x^2\right]} \cdot dx = \int x \cdot e^{u} \cdot dx.$$

Next I will use the formula $dx = \dfrac{du}{2 \cdot x}$ to substitute for dx in the indefinite integral, and then attempt to cancel out as many of the remaining x's as I possibly can.

$$\int x \cdot e^u \cdot dx = \int x \cdot e^u \cdot \frac{du}{2 \cdot x} = \int e^u \cdot \frac{du}{2} = \int \tfrac{1}{2} \cdot e^u \cdot du.$$

Fortunately, all of the x's did cancel out so $u = x^2$ was a reasonable choice for the "inside function." The indefinite integral that I have obtained is expressed entirely in terms of u and is less complicated than the one that we started out with. **Finding a formula for the antiderivative** is the second-to-last step in performing the technique of u-substitution, and this gives:

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The very last step in performing the technique of u-substitution is to use the definition of the inside function, $u = x^2$, to express the antidervative in terms of x:

$$\tfrac{1}{2} \cdot e^u + C = \tfrac{1}{2} \cdot e^{x^2} + C.$$

To summarize the work done in Question 5: we were asked to find the antiderivative for the function,

$$f(x) = x \cdot e^{x^2}.$$

The technique of u-substitution gave that the antiderivative was:

$F(x) = \tfrac{1}{2} \cdot e^{x^2} + C.$