Homework Assignment 15: Due at the beginning of class 4/3/02

The mathematical part of this homework involves setting up an integral. This integral will represent a volume of revolution (in this case, it will be formed by revolving a circle around the y-axis). If you set everything up correctly, then you should get an integral that is very difficult to evaluate using an antiderivative[1]. In the final parts of the homework, you will be asked to use your graphing calculator to estimate the value of the integral.

If you are under a lot of time pressure at the moment, skip to page 6 of this assignment.

Figure 1: Located near Harrisburg, PA, the Three Mile Island nuclear power station was the site of America's worst nuclear accident. Beginning at about 4am on Wednesday March 28, 1978, a combination of equipment failures and human errors resulted in the release of approximately 250,000 gallons of radioactive water from the plant and almost caused a melt-down of the reactor core.

During the course of the semester, you have made several calculations regarding energy and energy production. For example, during the class on "The Idea of an Antiderivative" (Monday March 11) you may have calculated that the world petroleum reserves will begin to run dry in approximately 30-50 years from now[2]. Likewise, you may also have read (see the ICE on global warming and greenhouse gases from Wednesday March 13) that other energy sources (such as coal and natural gas) may contribute to significant changes in the global climate and ecosystem such as rises in average global temperature and sea levels. Finally, you might have seen an ICE from very early on in the semester (Wednesday February 13) that describes some of the problems with another major energy source – the fission reactors that

[1] It is not impossible to evaluate the integral in this homework using antiderivatives. The integration technique that is required here is called *trigonometric substitution* and will probably be taught as part of Math 1b. You are not required to be familiar with the technique of *trigonometric substitution* in Math Xb, but if you would like to learn how to do it please fell free to visit Dr. Winter's office hours.

[2] You may also have calculated this result as a part of the final Math Xa lab: "Resource Wars."

generate nuclear power[3] (see Figure 1 [4]). The lab that you completed on the Chernobyl disaster may have brought home some of the risks inherent in the use of nuclear reactors to generate energy. As Homer J. Simpson (star of the hit TV show "The Simpsons") expressed the situation[5]:

> "…and thank you most of all for nuclear power, which is yet to cause a single proven fatality, at least in this country."

You might well wonder what the future of energy might be. The human race will certainly continue to need energy in the future, yet it seems as though our main sources of conventional energy are either rapidly dwindling or present potentially serious risks to health and the environment. Three alternative sources of energy are shown in Figure 2[6] (below). Although all have some impact on the environment (and in the case of poorly thought-out hydroelectric power schemes, this impact is sometimes severe), the potential for environmental damage and health risk are relatively small compared to current methods for generating large amounts of energy.

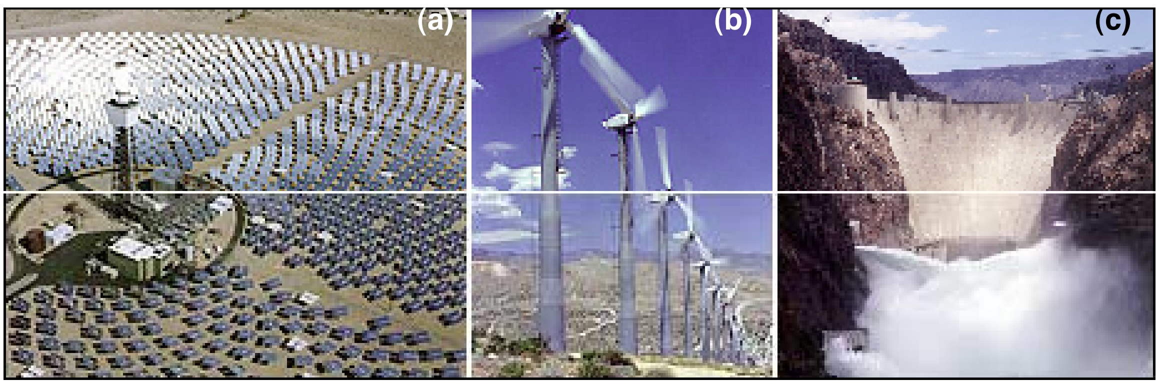

Figure 2: Alternative energy generation. (a) The experimental SOLAR-1 array near Barstow, CA. (b) Experimental wind turbine array located near Palm Springs, CA. (c) The non-experimental Hoover Dam, located approximately 30 miles south-west of Las Vegas, NV.

A fourth alternative to fossil fuels (coal, natural gas and petroleum) and fission-based nuclear reactors that has been the focus of a great deal of research is power generation by means of nuclear *fusion*.

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[3] The main problem described on the ICE from Wednesday February 13 was that conventional nuclear reactors produce highly toxic radioactive waste that cannot simply be dumped, and is both difficult and expensive to store.

[4] Image source: http://www.epa.gov/ If you are interested in learning more about the Three Mile Island accident, you can find a thorough description at: http://www.pbs.org/wgbh/amex/three/

[5] Homer recited this blessing before the Simpson family ate during episode 7F16 ("Oh brother, where art thou?") in which Homer discovers that he has a long-lost half brother (Herb Powell, played by Danny DeVito). Herb Powell runs a major car manufacturing firm in Detroit, which Homer quickly runs out of business with his design of a car "for the average man."

[6] Image sources are: (a) http://americanhistory.si.edu/csr/powering/images/alter1.htm and U.S. Department of Energy, Renewable Energy Laboratory
(b) http://americanhistory.si.edu/csr/powering/images/alter1.htm and U.S. Department of Energy, Renewable Energy Laboratory, and (c) http://www.howstuffworks.com/

Nuclear fission (currently used) involves the breaking apart of heavy atomic nuclei such as uranium to form smaller nuclei, energy and potentially harmful (both to humans and the environment) radiation. Figure 3[7] (which you might recognize from the review problems for Exam 1) shows the process of nuclear fission for the uranium isotope U-235 that is used in both nuclear power generation and in some nuclear weapons[8].

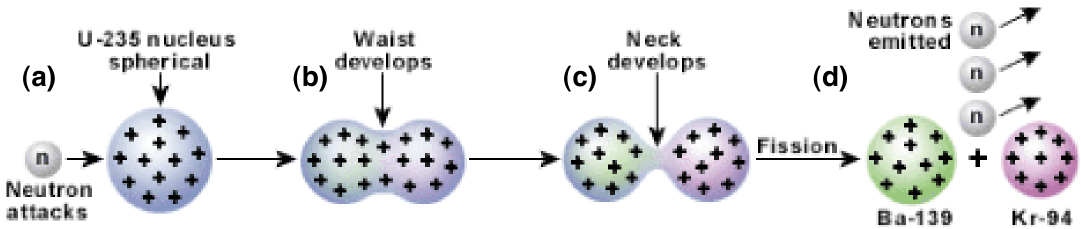

Figure 3: Steps in the process of nuclear fission. (a) A slow moving neutron strikes a uranium nucleus. (b) The imapct of the neutron makes the nucleus unstable, and the spherical shape of the nucleus changes. (c) The shape of the uranium nucleus is further deformed until the nucleus closely resembles two separate spheres joined by a narrow "neck." (d) The "neck" is severed creating two new nuclei (typically barium and krypton). As the "neck" is severed three neutrons and some energy are released. These neutrons will go on to collide with other nuclei (causing them to undego fission).

Nuclear fusion is the process by which small atomic nuclei are forced together under conditions of extreme pressure and temperature to form heavier nuclei (see Figure 4[9]). This process also releases energy and is a potential source of large-scale power generation for the future. (If you have ever seen the 1989 movie "Back to the Future II" starring Michael J. Fox and Christopher Lloyd, you might have noticed the "Mr. Fusion" unit that Lloyd's character uses to power his time-traveling Delorean coupe – see Figure 5[10]).

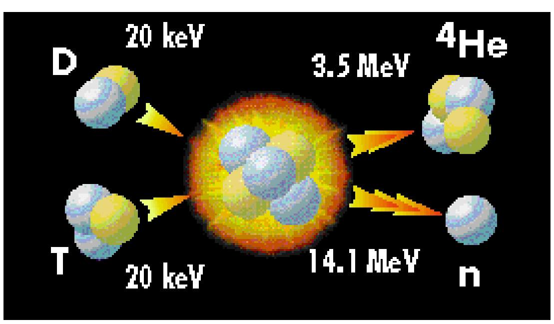

Figure 4: The nuclear fusion reaction favored by researchers into the use of nuclear fusion for large-scale power generation. Two isotopes of hydrogen (deuterium and tritium) are forced together under conditions of intense pressure and extreme temperature. The two hydrogen nuclei fuse to form one helium nucleus, one neutron and a considerable amount of energy.

Figure 5: In a scene from the 1989 blockbuster, "Back to the Future II" Doc Emmett Brown (played by Christopher Lloyd) refuels his Mr. Fusion power generator with household scraps.

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[7] Image source: http://www.education.eth.net/

[8] Typical modern thermonuclear weapons use some uranium-235 to get the explosion going. However, the critical fission reaction in a modern thermonuclear weapon is usually fission of plutonium. This fission supplies the heat and pressure needed required to initiate a fusion reaction in the beryllium "fuel" of the weapon. It is this fusion (small nuclei forced together to form larger nuclei) that generates most of the explosive power of a modern thermonuclear device, such as those carried by America's nuclear forces.

[9] Image source: http://FusEdWeb.llnl.gov/CPEP/

[10] Image source: http://

As is usually the case, Hollywood is far ahead of science in the arena of producing power from nuclear fusion. However, for the last decade researchers have been able to produce power from fusion reactions like the one illustrated in Figure 4. The very first time that humans generated energy by a controlled[11] fusion reaction was at 7:44pm on Saturday 9 November, 1991[12]. The facility used for this historic event was Joint European Torus (JET) located in Abingdon near Oxford, England (JET). The JET generated between 1.5 and 2 megawatts of power. In order to do so, the researchers had to heat a mixture of deuterium and tritium gas to about 200 million degrees Celsius. This is about 10 times hotter than the interior of the Sun.

The most powerful fusion reaction that has been reported so far[13] was carried out at the Tokamak Fusion Test Reactor (TFTR) located in the Princeton Plasma Physics Laboratory, near Princeton NJ (see Figures 6[14] and 7[15]).

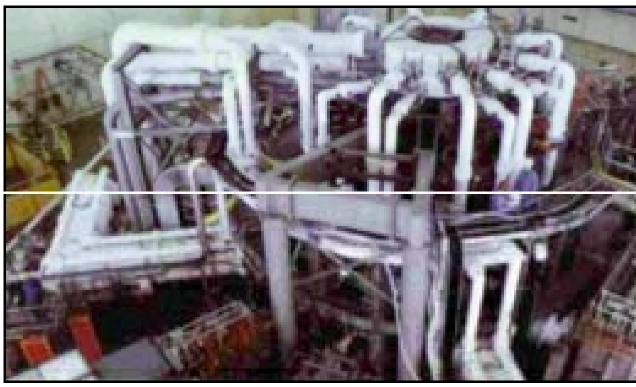

Figure 6: Exterior of the Tokamak Fusion Test Reactor (TFTR) located at the Princeton Plasma Physics Laboratory.

Figure 7: Interior of the TFTR. Note that the shape of the TFTR closely resembles the shape of a donut.

On May 10 1996, researchers were able to generate 10.7 MW of power and were able to sustain this reaction for 0.21 seconds. ("MW" stands for megawatts. For the sake of comparison, before the accident on the reactors at the Three Mile Island power plant were able to generate 1700 MW of electrical power.) Clearly, fusion power research has a long way to go before it becomes a viable means for supplying large numbers of people with the energy that they need.

Fusion power reactors like the TFTR work by heating a volume of deuterium and tritium (deuterium is the isotope ^{2}H of hydrogen and tritium is the isotope ^{3}H of hydrogen) gas to incredibly high temperatures until the electrons are liberated from the hydrogen atoms

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[11] Uncontrolled fusion reactions were first generated in the 1950's through atomic weapons research.

[12] Source: Joint European Torus Project Press Release, 1991.

[13] Source: R.J. Hawryluk. (1997) "Fusion plasma experiments on TFTR. A twenty year retrospective." Presented to the American Physical society 39^{th} Annual Meeting of the Division of Plasma Physics, November 17, 1997.

[14] Image source: http://www.plasmas.org/F-TFTRdevice.jpg

[15] Image source: http://www.physicscentral.com/

and the hydrogen passes to the fourth state of matter, a plasma[16]. Figure 8 [17] shows a plasma that was generated inside the TRTF in Princeton, NJ. The temperature of the plasma shown in Figure 8 is about 100 million degrees centigrade[18].

Figure 8: Plasma (superheated deuterium and tritium) inside the Tokamak Fusion Test Reactor.

(You probably have first-hand experience with plasma. If you have ever seen lightning, played with one of the small "plasma globes" sold in electronics and novelty stores or gazed at the Aurora Borealis – the Northern Lights – then you have seen a plasma with your own eyes.)

Once the hydrogen gas has been heated inside a fusion reactor, the plasma is confined to the reactor using strong magnetic fields (see Figure 9[19]). The main problems faced by researchers in plasma and fusion physics are keeping the plasma hot and keeping the plasma contained within the device.

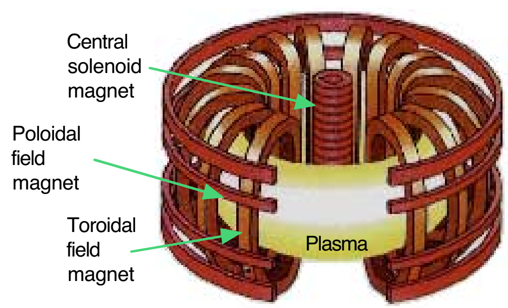

Figure 9: Scematic representation of a Tokamak fusion reactor. Note the three systems used to provide magnetic containment of the plasma: a central solenoid, a set of poloidal magnets that run around the outside of the device and a series of hoop-like toroidal magnets.

Up until now, researchers have only been able to sustain fusion reactions for a tiny fraction of a second (the run conducted on May 10 1996, for example, sustained a fusion reaction for about 0.21 seconds). In order to produce power for consumers, fusion reactors will have to sustain fusion reactions for long periods of time, or even operate on a continuous basis.

Advocates[20] of fusion as a large-scale source of energy are quick to point out some of the advantages that fusion reactors would have over other sources of energy. Some of these promised advantages include:

- **No risk of a nuclear accident.** According to advocates of fusion power, the amounts of deuterium and tritium use in fusion are so small that an uncontrolled release of energy (such as the "melt down" of a fission reactor) is impossible.

[16] The four states of matter are solid, liquid, gas and plasma.
[17] Image source: http://www.plasmas.org/fusion-mag.htm
[18] Source: http://www.pppl.gov/fusion_basics/pages/fusion_conditiond.html
[19] Image source: http://web.mit.edu/chungtk/www/fusion.html
[20] Source: http://www.pppl.gov/fusion_basics/

- **No air pollution.** As no fossil fuels (petroleum, coal) are used there is no release of harmful by-products (including greenhouse gases) into the atmosphere.
- **No high-level nuclear waste.** The only radioactive materials involved are deuterium and tritium. Although tritium is toxic to humans the amount required to produce fusion power is small compared with the amount of highly toxic radioactive waste associated with fission reactors.
- **No generation of weapons material.** Unlike the by-products of some commercial reactors (especially so-called "breeder" reactors that produce plutonium) the by-products of nuclear fusion do not appear to have applications to the construction of nuclear weapons.
- **Abundant fuel supply.** The main fuel component (deuterium) can be extracted from ordinary water. Estimates suggest that there is approximately 10^{16} kg of deuterium in the surface waters of the earth[21].

As this list suggests, many of the benefits associated with using nuclear fusion as a source of energy accrue from the small amounts of radioactive material needed to generate power, and the abundance of this fuel source. In this homework assignment, you will calculate how much deuterium and tritium will be required to fuel a fusion reactor with the same power output as the fission reactors at the Three Mile Island nuclear power station.

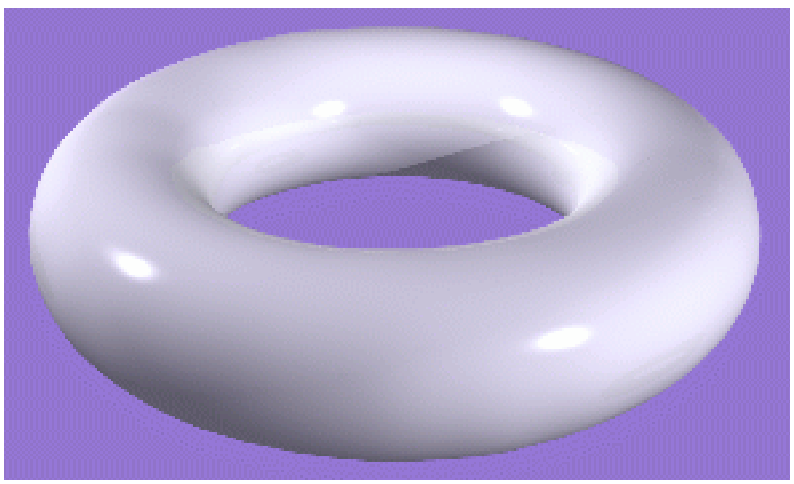

Figure 10: A solid torus. This shape can be generated by taking a circle and revolving the circle around the y-axis.

In a fusion reactor like the TFTR, the plasma is confined in a shape called a solid torus (see Figures 8, 9 and 10[22]). This is a shape like a donut or bagel. You can imagine that a solid torus is formed by taking a circle and revolving that circle around the y-axis (see Figure 11).

In Questions 1, 2 and 3 of this homework assignment you will set up an integral that gives the volume of a solid torus. In Question 4 of this homework assignment you will estimate the numerical value of this integral with your calculator and in Question 5 of the homework assignment you will use the results of these calculations to determine how much fuel a fusion power plant would require on a daily basis.

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[21] For the sake of comparison, the most generous estimates of the world's remaining reserves of petroleum suggest that there are at most about 5×10^{14} kg of oil remaining. (Source: BP-Amoco. "Statistical Review of World Energy." 2000.)

[22] Image source: http://www.me.washington.edu/~cdam/GALLERY/images/CHEK/torus.html

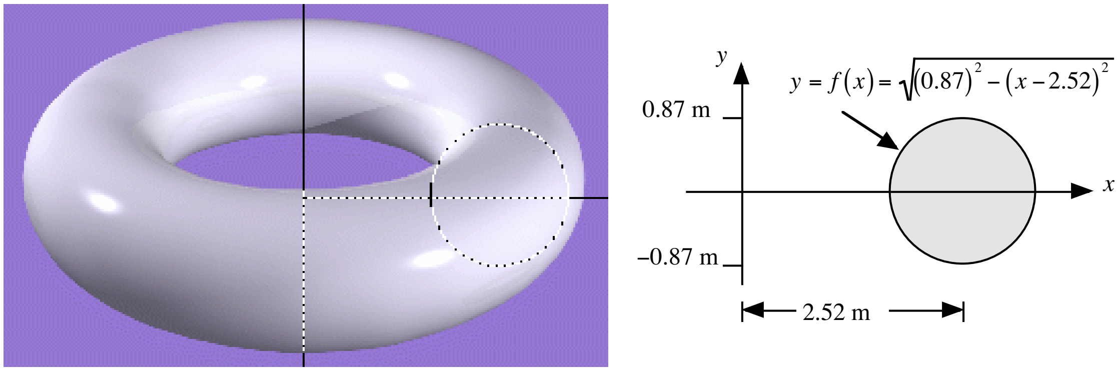
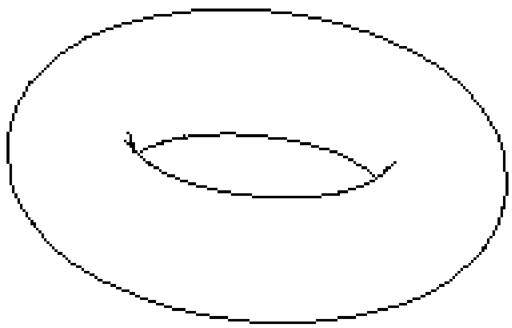


Figure 11: A solid torus can be thought of as the shape created when a circle (such as the one shown) is revolved around the y-axis.

1. Your first job in setting up the integral for the volume of the solid torus is to break the solid torus up into regular shapes whose volume you can calculate. Using the diagram[23] given below, sketch one of the regular shapes that you will break the solid torus into.



NOTE: You should hand in this diagram (with your sketch) or a copy as your answer to Question 1.

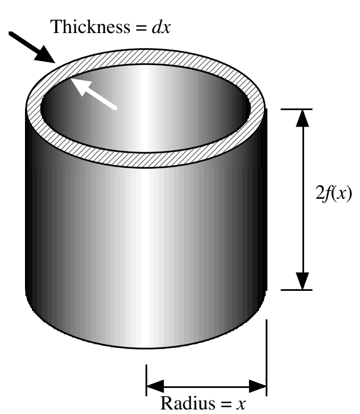


Figure 12: The thick-walled cylinder.

2. Figure 12 shows a thick-walled cylinder. The radius of the (empty) interior of the cylinder is x. The thickness of the wall of the cylinder is dx and the height of the cylinder is $2 \cdot f(x)$. Find a formula (involving x, dx and $2 \cdot f(x)$) for the volume of the walls of the cylinder.

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[23] Image source: http://www.neci.nec.com/homepages/zliu/line-drawings.html

3. The function that describes the top half of the circle in Figure 11 is described by the equation:

$$y = f(x) = \sqrt{(0.87)^2 - (x - 2.52)^2}\,.$$

The numbers shown in Figure 11 (0.87 meters, 2.52 meters) are the actual numbers for the solid torus of plasma that was used to generate 10.7 MW of power in the Tokamak Fusion Test Reactor (TFTR) on May 10, 1996. Use these numbers, together with the equation given for $f(x)$ and your answers to Questions 1 and 2, to set up an integral that will give the volume of the plasma that was used to generate 10.7 MW of power. You integral should include:

- A numerical value for the lower limit of integration.
- A numerical value for the upper limit of integration.
- An explicit formula for the function that is being integrated.

4. Use the "sum" and "seq" commands on your graphing calculator to estimate the value of the integral from Question 3. Use 100 rectangles to make the estimate.

5. According to researchers[24] at the Princeton Plasma Physics Laboratory, when the TFTR produced 10.7 MW of fusion power, the density of the plasma in the reactor was about 1.4×10^{-9} kg per cubic meter. Use your answer to Question 4 to calculate the amount of plasma (in units of kg) that was in the reactor when the 10.7 MW of power were generated. This is the amount of plasma[25] that was required to generate 10.7 MW of power for a total of 0.21 seconds. Calculate the amount of plasma (in units of kg) that would be required to generate 1700 MW of power[26] for a total of 24 hours.

Epilogue

By way of comparison, a 1700 MW coal-fired power station will burn about 15.3 million kg of coal on a daily basis. This produces about 51,000 metric tons of the greenhouse gas carbon dioxide (CO_2) and about 140 metric tons of the greenhouse gas nitrous oxide (NO_2) on a *daily* basis. In addition, the burning of this much coal produces about 1000 metric tons of sulfur dioxide (SO_2) on a *daily* basis. Sulfur dioxide dissolves in atmospheric water to produce highly corrosive and poisonous precipitation (a mixture of H_2SO_3 and H_2SO_4) popularly called acid rain.

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[24] K.M. McGuire et al. (1995) "Review of D-T results from TFTR." *Physics of Plasmas*, **2**(6): 2176-2202.

[25] This is also the amount of hydrogen (deuterium and tritium) fuel that the reactor needed to generate 10.7 MW of power for 0.21 seconds.

[26] This is the power output that the two reactors at the Three Mile Island fission plan were capable of generating before the accident in 1978.