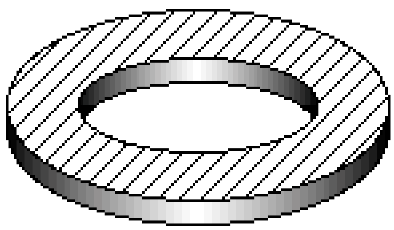
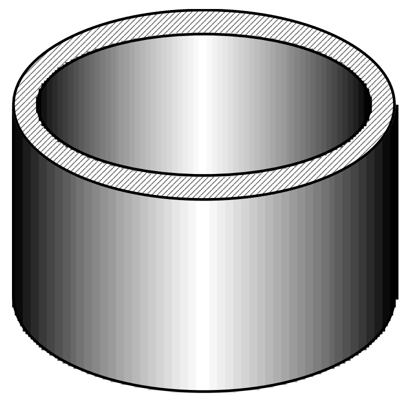
Homework Assignment 15: Solutions

1. There are many ways to break the solid torus shape up. Some of the ways of breaking up the solid torus will yield shapes whose volumes are reasonably easy to calculate, whereas other schemes for breaking up the solid torus will produce shapes whose volume will be very difficult to calculate.

Two shapes that are worth trying are thick-walled cylinders and flat washers (see below).

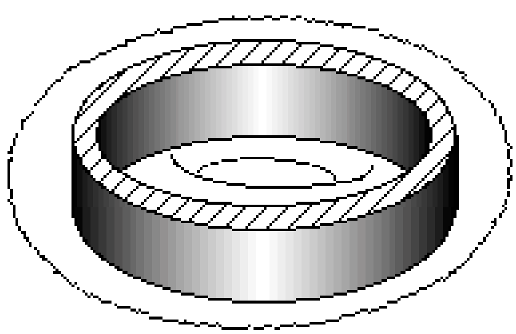


A flat washer



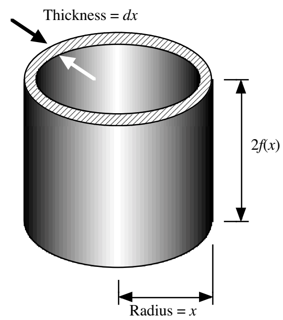
A thick walled cylinder

In this particular set of solutions we will break the solid torus up into thick-walled cylinders mainly because this will produce an integral with the familiar x as the variable[1]. The outline of the solid torus with the thick walled cylinder sketched in is shown below.



[1] If you use the flat washers then the integral that you set up should have y as the variable. Although such an integral is (in principle) no more difficult to evaluate than an integral which has x as the variable, people sometimes find the use of y as a variable confusing.

2. The thick walled cylinder is shown below.

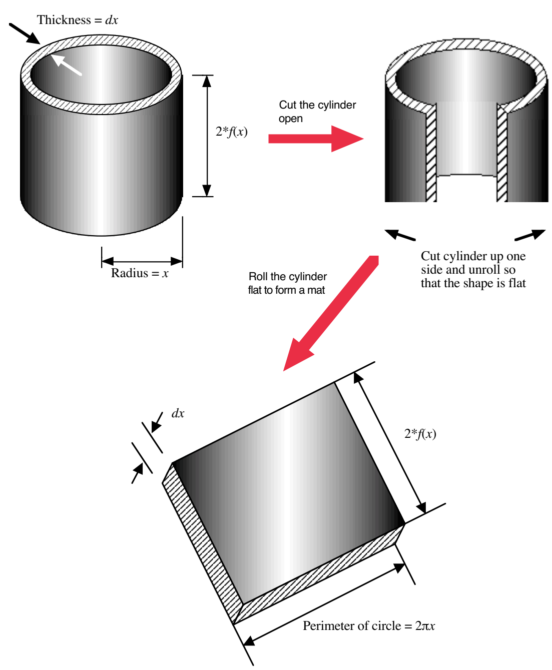


The formula for the volume of this thick walled cylinder is:

$$Volume \;=\; (2\pi\cdot x)\cdot(2\cdot f(x))\cdot dx = 4\pi\cdot x\cdot f(x)\cdot dx$$

To see how this volume formula was obtained, consider the manipulations shown in the diagram below. First, imagine taking the cylinder and cutting it open with a pair of scissors. Once the cylinder has been cut, imagine it being opened out and rolled flat to form a rectangular mat. The thickness of that rectangular mat is the thickness of the walls of the cylinder, dx. The width of the mat is the height of the cylinder, $2\cdot f(x)$. The length of the mat is the perimeter of the circle at the top of the cylinder (which has radius x), $2\pi x$. The volume of the rectangular mat is given by:

$$Volume \;=\; (Thickness)\cdot(Width)\cdot(Length) \;=\; (dx)\cdot(2\cdot f(x))\cdot(2\pi\cdot x).$$



3. To set up the integral for the volume of the we have to determine the limits of integration and an explicit formula for the function $f(x)$. Fortunately the formula for $f(x)$ was given on the homework assignment, and it is:

$$f(x) = \sqrt{(0.87)^2 - (x - 2.52)^2}.$$

The limits of integration can be deduced from Figure 11 of the homework assignment (shown below).

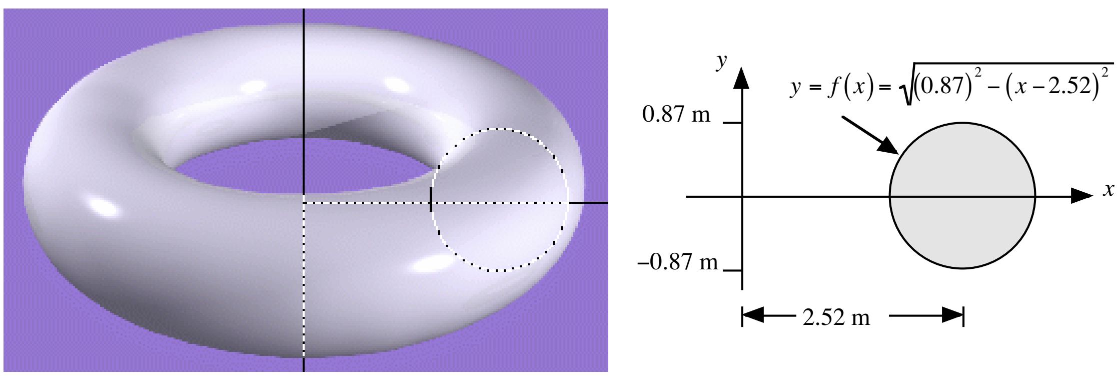
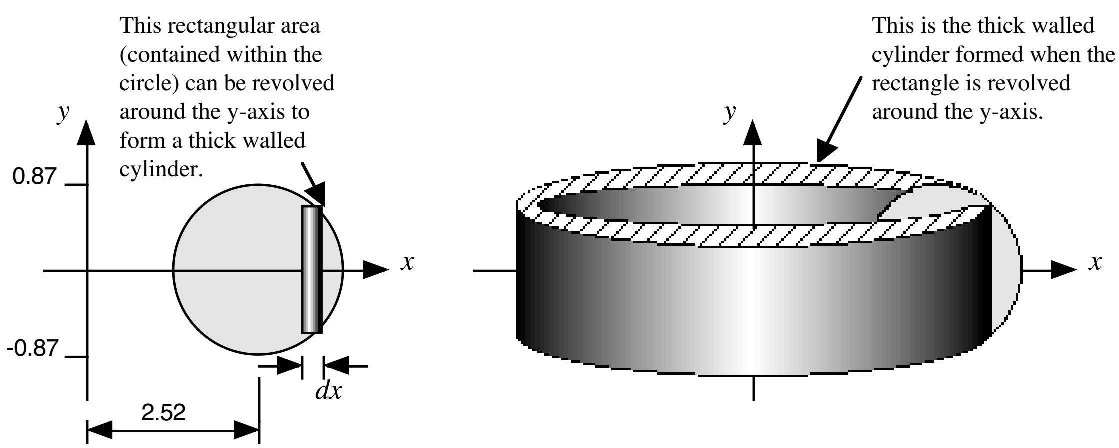


Figure 11: A solid torus can be thought of as the shape created when a circle (such as the one shown) is revolved around the y-axis.

One way to imagine the construction of the solid torus is to take the small gray circle shown in Figure 11 and revolve this circle around the y-axis. If you imagine taking a thin rectangular portion of the gray circle and revolving this around the y-axis then the volume that would be created is the volume of one of the thick walled cylinders that we are breaking the solid torus into, as shown in the diagram given below.



Integrals work a bit like sums – they add together the areas (or in this case volumes) of things like rectangles when the thickness of the rectangle, dx, approaches a limit of zero. In a regular integral where you are trying to work out the area under a curve, the lower limit of integration is the x-value where the little rectangles begin and the upper limit of integration is the x-value where the little rectangles end.

For the integral that we are trying to set up, the upper and lower limits of integration can be determined in the same way. The x-value where the little rectangles (i.e. the rectangles that are revolved around the y-axis to form the thick walled cylinders) begin at the left-most edge of the gray circle and they end at the right-most edge of the gray circle. The center of the gray circle is at $x = 2.52$ and the radius of the gray circle is 0.87. The x-coordinates of the left-most and right-most edges of the gray circle are therefore:

$$\textit{x-coordinate of left-most edge} = 2.52 - 0.87 = 1.65$$

$$\textit{x-coordinate of right-most edge} = 2.52 + 0.87 = 3.39$$

The integral that gives the total volume of the solid torus will be:

$$\textit{Total volume of solid torus} = \int_{Left\_Edge}^{Right-Edge} (Volume\_thick\_wall\_cylinder)$$

$$= \int_{1.65}^{3.39} 4\pi \cdot x \cdot \sqrt{(0.87)^2 - (x - 2.52)^2} \cdot dx$$

4. The relevant calculator commands (for a TI-83 calculator) to estimate the integral from Question 3 using 100 rectangles are:

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Y1 = 4*π*X*√((0.87)^2 – (X-2.52)^2)

(3.39 – 1.65)/100 STO⇨ W

sum(seq(Y1(1.65 + K*W)*W, K, 0, 99))
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The result that you should obtain (using 100 rectangles) is that the integral is approximately equal to:

$$\textit{Volume of solid torus} \approx 37.61049191.$$

This means that the volume of plasma[2] in the TFTR on 10 May 1996 was approximately equal to 37.61049191 cubic meters.

[2] The exact value of the integral is 37.65033, so the estimate from your calculator is quite accurate.

5. Finally, we will perform the calculation of the amount (in units of kg) of plasma required to run a 1700 MW fusion power station for 24 hours. We are going to do this in three steps:

a) First we will calculate the amount of plasma (in kg) that was required for the TFTR to generate 10.7 MW of power for 0.21 seconds.
b) Second we will convert this answer to the amount of plasma (in kg) that would be required for the TFTR to produce 1700 MW of power for 0.21 seconds.
c) Third we will convert the second answer to calculate the amount of plasma (in kg) required for the TFTR to produce 1700 MW of power for 24 hours.

Step 1: Amount of plasma for 10.7 MW lasting for 0.21 seconds

According to the Princeton Plasma Physics Laboratory, the density of the plasma during the fusion reaction was about 1.4×10^{-9} kg per m^3. As you may have learned in a chemistry or physics class, mass volume and density are related by the equation:

$$Mass = (Density)\cdot(Volume).$$

Using the figure of 1.4×10^{-9} kg per m^3 for the density and the figure of 37.61049191 m^3 for the volume gives the mass of plasma:

$$\textit{Mass of plasma} = 5.265468867\times10^{-8} \text{ kg}.$$

Step 2: Amount of plasma for 1700 MW lasting for 0.21 seconds

Here we will multiply the previous answer by a factor of $\frac{1700MW}{10.7MW}$. This gives the mass of plasma as:

$$\textit{Mass of plasma} = 8.365698201\times10^{-6} \text{ kg}.$$

Step 3: Amount of plasma for 1700 MW lasting for 24 hours

The previous answer is the amount of plasma required for 0.21 seconds of operation. To sustain the fusion reaction for one full second, about $\frac{1\sec}{0.21\sec}\approx 4.7619$ times as much plasma would be required. This mass of plasma would be:

$$\textit{Mass of plasma for one full second} \approx 3.983661826\times10^{-5} \text{ kg}.$$

In twenty four hours, there are 24*60*60 = 86,400 seconds. To obtain the mass of plasma required to generate 1700 MW for twenty four hours, we can multiply the mass of plasma required for one full second of power generation by 86,400. Doing this gives:

$$\textit{Mass of plasma for 24 hours} \approx 3.441883818 \text{ kg}.$$

This is about the same as the mass of a gallon of milk.