Homework Assignment 16: Solutions

The mathematical content of this assignment consisted of calculating average values of functions using areas under curves, integration and antiderivatives. In some calculus course, only the formulaic approach to average value is emphasized. By "formulaic approach" I mean an introduction to the idea of average value that goes no deeper than that presented in the first paragraph of Homework 16:

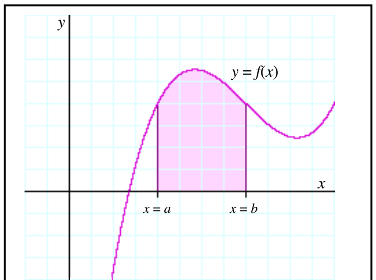

Figure 1: The function and shaded area between *a* and *b*.

Figure 1 shows the graph of a function $y = f(x)$. The area beneath the curve between $x = a$ and $x = b$ has been shaded. Recall that the average value of the function $f(x)$ between $x = a$ and $x = b$ is given by:

$$\text{Average value} = \frac{\int_a^b f(x)\cdot dx}{b-a}.$$

The intention of many of the problems on Homework 16 was to get you to work with the idea of average value in a non-formulaic (possibly intuitive) way.

1. Figure 6 (from the homework assignment) is reproduced below.

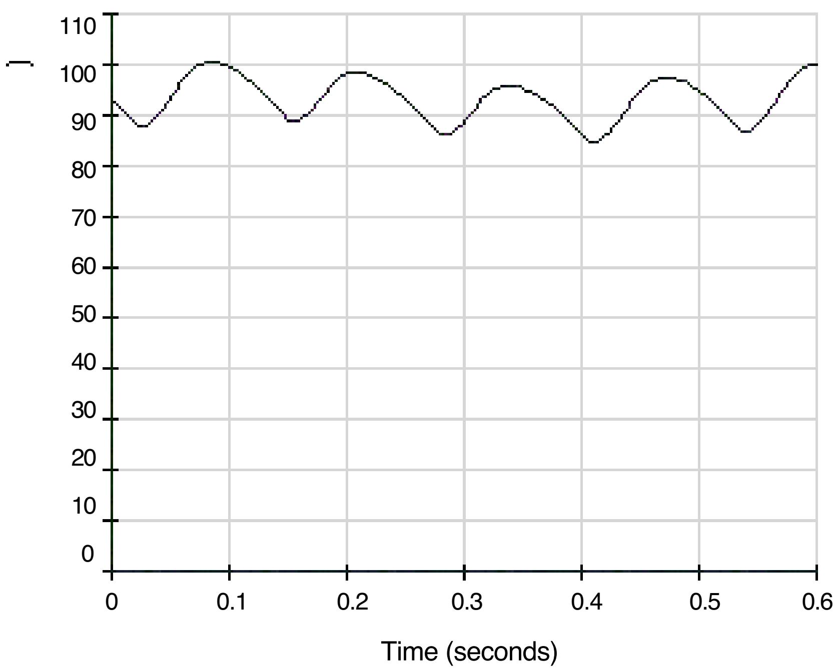

Figure 6: Graph showing blood pressure in neck arteries versus time for one of the rhesus monkeys on board the Cosmos 1514 satellite.

There are many ways that you could estimate the area under this curve between $T = 0$ and $T = 0.6$. Two possibilities include:

- Try to find a function, say $b(T)$, whose graph closely matches the curve shown in Figure 6. Find an equation for the antiderivative, say $B(T)$, for this function and use this antiderivative to calculate the area by evaluating: $B(0.6) - B(0)$.
- Break the area under the curve in Figure 6 up into a collection of regular shapes such as squares, rectangles and triangles. Use area formulas from geometry to calculate the area of each and total all of these small areas to obtain an estimate for the total area under the curve between $T = 0$ and $T = 0.6$.

The first approach would be quite feasible if we had already studied trigonometry. A trigonometric function whose graph matches the curve in Figure 6 quite closely is given by[1]:

$$b(T) = -5.714 \cdot \sin\left(\tfrac{2\pi}{0.128} \cdot T\right) + 92.143.$$

An antiderivative of this function is given by:

$$B(T) = 0.1164 \cdot \cos\left(\tfrac{2\pi}{0.128} \cdot T\right) + 92.143 \cdot T + C,$$

where C is a constant (whose value is not known). Using this antiderivative to calculate the area under the curve in Figure 6 gives[2]:

Area under curve = $B(0.6) - B(0) \approx 55.125$ mmHg·seconds.

The approach that I expected you to use was to break the area under the curve shown in Figure 6 into regular shapes such as rectangles and triangles. One way of breaking up the area beneath the curve in Figure 6 (along with the areas of each of the regular shapes used) is shown on the next page. The sum total of all of the areas shown is:

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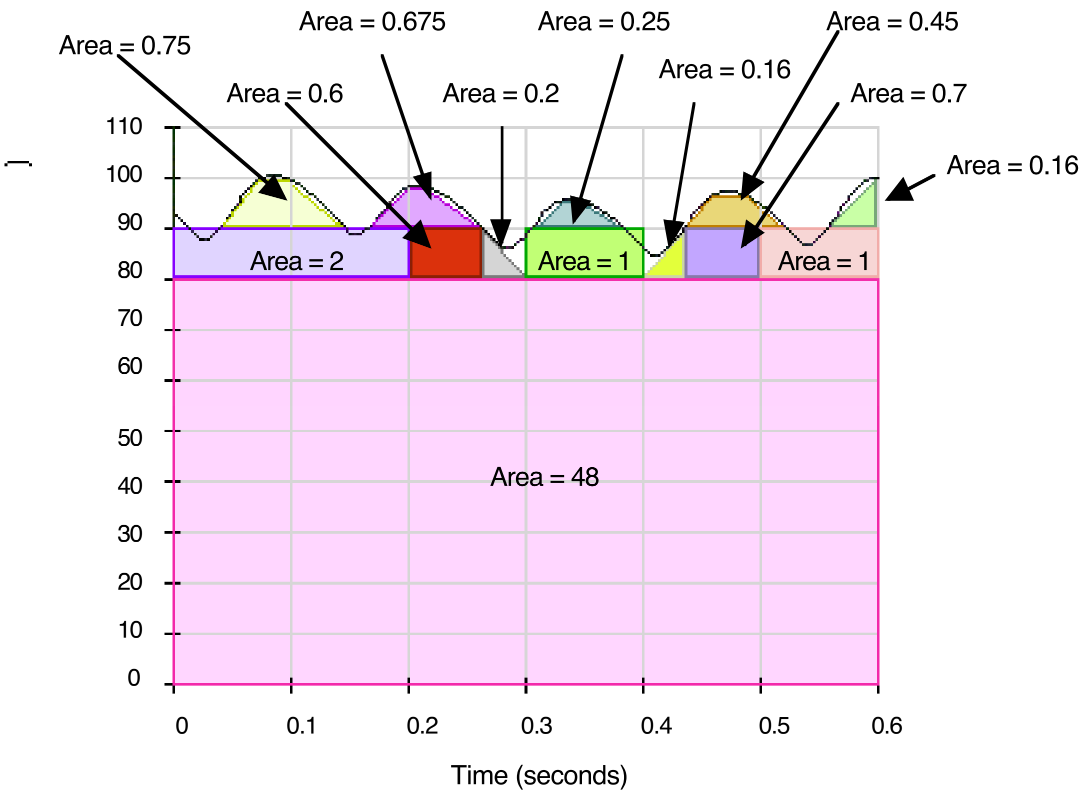
[1] Please note that I am giving this equation (and the subsequent calculations) to demonstrate how they would be done and perhaps to whet your appetite for the some of the fruits that our virtuous and trigonometric labors will present over the course of the next few weeks. I do not expect you to be able to fit trigonometric functions to data at this point of the course, nor are you expected to be able to find derivatives and antiderivatives of trigonometric functions at present.

[2] When evaluating the trigonometric functions (such as sin, cos and tan) it is important to make sure that your calculator is in RADIAN mode. When you graph a trigonometric function it is also a good idea to make sure that your calculator is set to RADIAN mode.

Area under curve ≈ 48 + 2 + 0.75 + 0.675 + 1 + 0.25 + 0.45 + 0.16 + 1 + 0.7 + 0.16 + 0.6 + 0.2

≈ 55.945 mmHg·seconds.



2. The average blood pressure in the arteries in the neck of the rhesus monkey can be determined by dividing the area under the curve in Figure 6 by the width of the area. The width of the area is 0.6 – 0 = 0.6. Using the estimated area under the curve obtained by breaking the area into rectangles and triangles gives:

$$\textit{Average blood pressure} = \frac{55.945}{0.6} = 93.241 \text{ mmHg}.$$

According to the information provided in the homework assignment, when a rhesus monkey is on Earth, the blood pressure in the neck arteries varies between 60 and 80 mmHg, with an average of 72 mmHg. The average obtained here (93.241 mmHg) is considerably higher than 72 mmHg, supporting the hypothesis that when exposed to a microgravity environment, the blood pressure in the head and upper body of a primate increases.

3. What does the previous calculation have to do with the formula for average value:

$$Average\ value = \frac{\int_a^b f(x)\cdot dx}{b-a}\ ?$$

We calculated the average blood pressure of the rhesus monkey by dividing the area under a curve by the width of that area. The area under the curve in Figure 6 was 55.945 mmHg·seconds. The width of the area in Figure 6 was 0.6 seconds.

If we said that the function shown in Figure 6 was named $f(x)$, and that $a = 0$ and $b = 0.6$, then another way to represent the area under the curve in Figure 6 would be with the integral notation:

$$Area\ under\ curve = \int_0^{0.6} f(x)\cdot dx = \int_a^b f(x)\cdot dx\,.$$

The width of the interval was $0.6 = 0.6 - 0 = b - a$.

Using these symbols, the calculation that we did to determine the average blood pressure of the rhesus monkey will look like this:

$$Average\ blood\ pressure = \frac{\text{Area Under Curve}}{\text{Width of Area}} = \frac{\int_a^b f(x)\cdot dx}{b-a},$$

which is exactly the same as the formula for calculating the average value of a function.

4. Using the variables:

x = Distance from Japan (in units of thousands of kilometers)
y = Depth of Pacific Ocean (in units of kilometers)

the shape of the Pacific Ocean floor was approximated by the equation:

$$y = p(x) = 0.048\cdot x^4 - 0.9\cdot x^3 + 5.79\cdot x^2 - 14.89\cdot x + 7.69\,.$$

An antiderivative, $P(x)$, of this function is given by:

$$P(x) = \tfrac{0.048}{5}\cdot x^5 - \tfrac{0.9}{4}\cdot x^4 + \tfrac{5.79}{3}\cdot x^3 - \tfrac{14.89}{2}\cdot x^2 + 7.69\cdot x + C\,.$$

If we were to evaluate the quantity $P(9) - P(0.7)$ then you would calculate the area between the curve and the x-axis. The curve in this problem is the ocean floor and the x-axis in this problem is the surface of the ocean. The area that is between the surface of the ocean and the ocean floor is the dark blue area from Figure 7 on the homework assignment.

5. The average depth of the Pacific Ocean will be given by:

$$\textit{Average depth of Pacific Ocean} = \frac{\text{Dark blue area from Figure 7}}{\text{Width of dark blue area from Figure 7}}$$

$$= \frac{\int_{0.7}^{9} p(x)\cdot dx}{9-0.7}$$

$$= \frac{P(9)-P(0.7)}{9-0.7}.$$

Evaluating the antiderivative $P(x)$ at $x = 0.7$ and $x = 9$ gives:

$$P(0.7) = 2.344530972 + C.$$

$$P(9) = -36.2196 + C.$$

Substituting these values into the expression for the average depth of the Pacific Ocean gives:

$$\textit{Average value} = \frac{-38.56413097}{8.3} = -4.64628084 \text{ km}.$$

So, according to these calculations, the average depth of the Pacific Ocean is approximately 4.6 kilometers. Most sources give the average depth of the Pacific Ocean as about 4.2 kilometers. Our result is remarkably accurate given that it was based on a very rough picture (Figure 7 on the homework assignment) picture of the ocean floor, and an even rougher approximation (the ocean floor is definitely a much more complicated curve than the simple quartic, $p(x)$, given in Homework 16).