Homework Assignment 18: Solutions.

1. When you are slicing up the volume of a lake to find the total amount of fish biomass in the lake, the equation that you use is:

(*Amount of fish biomass*) = (*Density of biomass*)×(*Volume*).

However, this equation is only appropriate to use when the density is constant (or nearly constant). When the value of the density varies a great deal, simply multiplying the value of the density at one particular point of the lake by the volume of the lake will almost certainly not give an accurate figure for the total amount of fish biomass in the lake.

In order to calculate the total fish biomass in the lake when the density is not constant, the approach is to:

i. Break up the volume of the lake into smaller pieces. The shape of each of these smaller pieces should be chosen that throughout each small piece, the value of the density function is very nearly constant.
ii. Calculate the amount of fish biomass in each small piece by multiplying the value of the density function (evaluated at some point within the small piece of lake) by the volume of the small piece of the lake.
iii. Add all of these fish biomasses together to get the total amount of fish biomass in the whole lake.

Note that the "adding together" part of this process is normally done with an integral.

The key to deciding how to break the volume of the lake up into smaller pieces is the algebraic structure of the density function. In this case, the density function, $p(x)$, was defined by the equation:

$$p(x) = -188323018.5\cdot x^4 + 14714825.03\cdot x^3 - 368440.0735\cdot x^2 + 3174.116\cdot x + 0.1$$

where x represents the depth in units of kilometers. We have to break the volume of the lake up into pieces that will keep the value of the density function, $p(x)$, very close to being constant. Since the density function $p(x)$ only depends on the variable x (depth), the shapes that keep the value of x very nearly constant will also keep the value of the density function $p(x)$ very nearly constant. As the variable x represents the depth, the shapes that we should break the lake into are the shapes that have very nearly constant depths – that is, thin horizontal slabs as shown in Figure 1.

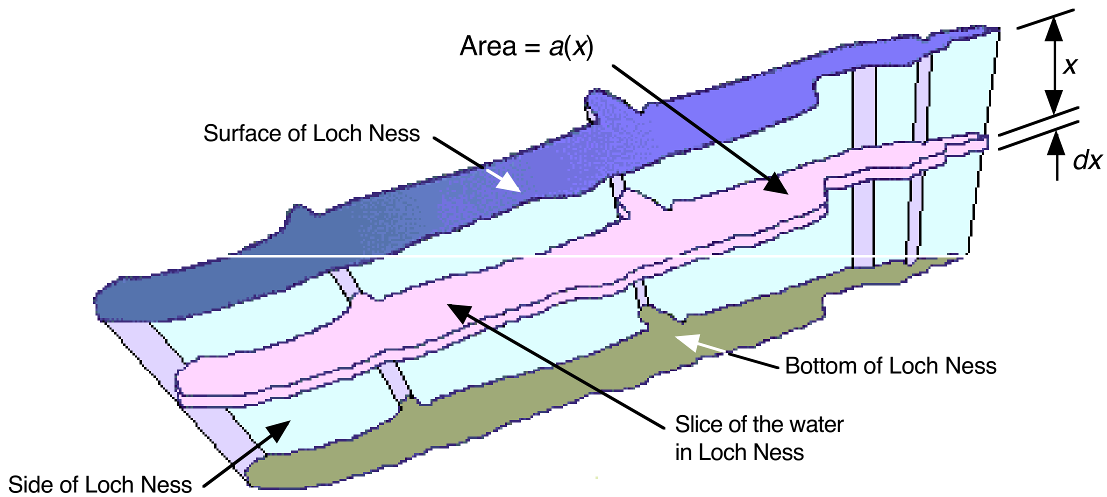

Figure 1: Three-dimensional model of Loch Ness. The pink slice in the middle of the diagram is at a depth of *x*, has an area of *a*(*x*), and a thichness of *dx*.

2. The volume of each of the horizontal slabs that Loch Ness is sliced into (see Figure 1) is given by:

$$(\textit{Volume of horizontal slab}) = (\textit{Area of top of slab})\times(\textit{Thickness of slab})$$

$$= a(x)\cdot dx$$

$$= 58.88\cdot(1-x)\cdot dx$$

The quantity that we are trying to determine is the total amount of fish biomass in Loch Ness. The quantity that we will actually integrate is the amount of fish biomass in each of the horizontal slabs that Loch Ness is sliced into. To obtain a formula for the amount of fish biomass in each horizontal slab:

$$(\textit{Amount of fish biomass in Slab}) = (\textit{Density of biomass})\times(\textit{Volume})$$

$$= p(x)\cdot a(x)\cdot dx$$

$$= p(x)\cdot 58.88\cdot(1-x)\cdot dx$$

The limits of integration will be the shallowest depth at which fish are found (for the lower limit of integration) and the deepest depth at which substantial numbers of fish are found (for the upper limit of integration). According to the homework assignment, fish were found between the surface and a depth of 40 meters, with very few fish below 40 meters. Expressing these depths in units of kilometers gives the limits of integration as:

- Lower limit of integration: 0 (Surface of the Loch)
- Upper limit of integration: 0.04 (Depth of 40 meters)

The integral giving the total fish biomass in the Loch (in units of metric tons will be:

$$\textit{Amount of fish biomass} = \int_0^{0.04} p(x)\cdot a(x)\cdot dx.$$

3. In order to find the total amount of fish biomass in Loch Ness, we evaluate the integral from Question 2. To do this, we must find an antiderivative for the quintic (degree 5) polynomial given by:

$$p(x)\cdot a(x) = (1.109\times10^{10})\cdot x^5 - (1.195\times10^{10})\cdot x^4 + (8.881\times10^8)\cdot x^3 - (2.188\times10^7)\cdot x^2 + (1.869\times10^5)\cdot x + 5.888$$

The antiderivative is found using the standard antidifferentiation rule for power functions applied on a term by term basis. Naming this antiderivative $F(x)$, we get:

$$F(x) = (1.109\times10^{10})\cdot\frac{x^6}{6} - (1.195\times10^{10})\cdot\frac{x^5}{5} + (8.881\times10^8)\cdot\frac{x^4}{4} - (2.188\times10^7)\cdot\frac{x^3}{3} + (1.869\times10^5)\cdot\frac{x^2}{2} + 5.888\cdot x + C$$

The amount of fish biomass in Loch Ness is, then,

$$\begin{aligned}\textit{Amount of fish biomass} &= \int_0^{0.04} p(x)\cdot a(x)\cdot dx \\ &= F(0.04) - F(0) \\ &= 14.20096 \text{ metric tons.}\end{aligned}$$

Therefore, the amount of fish biomass in Loch Ness is approximately 14 metric tons.

4. The objective of this question was to create two equations. The first equation was to give daily food consumption as a function of body mass, and the second equation was to give daily food consumption as a function of body length.

Daily Food Consumption as a Function of Body Mass

Entering the data into a calculator and producing a STATPLOT gives the screen shown as Figure 2 (below).

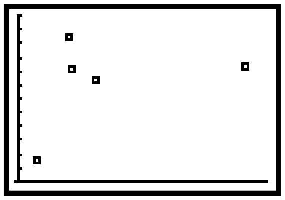

Figure 2: STAT-PLOT showing food versus body mass.

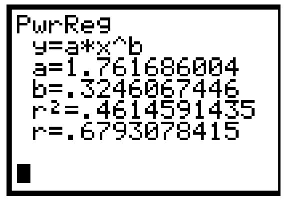

Figure 3: Results of power regession with food consumption and body mass.

The STATPLOT does not reveal any particular clear pattern, so any kind of relationship that does a reasonable job of matching the data could be used. We used power regression (see Figure 3) simply because the STATPLOT bore a slight resemblance to the graph of a power function with a power between zero and one.

Using M to denote body mass (in units of kilograms) and $f(M)$ to denote the daily food consumption (in units of kilograms), power regression on a calculator gave the following equation to represent this relationship:

$$f(M) = 1.7617 \cdot M^{0.3246} \ .$$

Daily Food Consumption as a Function of Body Length

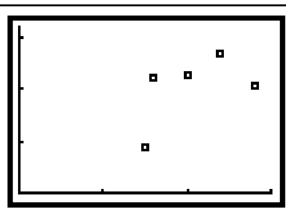

Figure 4: STAT-PLOT showing food versus body length.

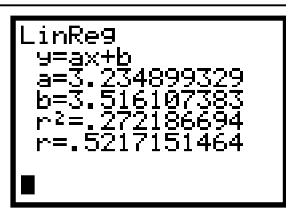

Figure 5: Results of linear regession with food consumption and body length.

Entering the data into a calculator and producing a STATPLOT gave the screen shown in Figure 4. As before, there was no particularly distinct or compelling pattern revealed by the STATPLOT. We performed a linear regression on the data as a linear function would probably do as good a job of representing the trend in this data as any function. (The results of this regression are shown in Figure 5.)

Using L to denote body mass (in units of meters) and $g(L)$ to denote the daily food consumption (in units of kilograms), linear regression on a calculator gave the following equation to represent this relationship:

$$g(L) = 3.2349 \cdot L + 3.5161 \ .$$

5. The functions $f(M)$ and $g(L)$ can be used to compute the daily food consumption of each of the creatures listed in Table 2 of the homework assignment. The results of this are shown in the table below.

Common name	Scientific name	Typical length (meters)	Typical mass (kg)	Daily food consumption from mass (kg)	Daily food consumption from length (kg)
Elasmosaurus	*Woolungo-saurus glendowrensis*	13.7 (body and neck) 5.7 (body)	2320	21.79	47.83 (body and neck) 21.96
Rotting carcass recovered by Zuiyo-Maru	Unknown	9.6	1800	20.07	34.57
Nessie	*Nessiteras rhambopteryx*	4.5	Unknown	Unknown	18.07

The estimates of the daily food requirements of these creatures range from a low of 18.07 kg of biomass per day to a high of 47.83 kg of biomass per day.

According to the article by Sheldon and Kerr[1], 10-20 large predatory animals would be required to sustain a viable breeding population. Assuming that these animals were piscivorous (fish-eating), the daily food requirements of such a group of animals ranges from a minimum of 180.7 kg of fish biomass per day (10 creatures, eating 18.7 kg each) to 956.6 kg of fish biomass per day (20 creatures, each consuming 47.83 kg).

At this rate, the 14.20096 metric tons (14200.96 kg) of fish biomass in Loch Ness would be completely consumed by the population of creatures in 15 ($\approx \frac{14200.96}{956.6} = 14.8452$) to 79 ($\approx \frac{14200.96}{180.7} = 78.5886$) days. Although these numbers are all estimates and projections, the situation that they suggest is that there simply is not enough food available in the form of fish biomass in Loch Ness to support a viable breeding population of large aquatic predators for any length of time. Unfortunately for Nessie enthusiasts, these numbers do not support the existence of a large, unknown, fish-eating aquatic animal in the Loch.

[1] R.W. Sheldon and S.R. Kerr. (1972) "Density of monsters in Loch Ness." *Limnology and Oceanography*, **17**: 746-798.