Homework Assignment 1: Solutions

When reading these solutions, remember that phrases such as “number of heroin users” or “number of heroin addicts” are used with the meaning described below.

> Suppose you took a random sample of 100,000 high school seniors from across the United States. The “number of heroin users” is the number of students from this group of 100,000 who used heroin during the previous 30 days.

1. The completed version of Table 1 is shown below.

Year	Number of heroin users	Rate of change	Amount that number of users will change by in 5 years	New number of heroin users
1975	400	–40	–200	200
1980	200	20	100	300
1985	300	–20	–100	200
1990	200	80	400	600
1995	600	–33.3	–166.5	433.5

Table 1

The very last entry in the table gives the (approximate) number of high school seniors using heroin five years after 1995. Therefore, the (approximate) number of high school seniors using heroin in the year 2000 was about 433.5.

2. The completed version of Table 2 is shown below.

Year	Number of heroin users	Rate of change	Amount that number of users will change by in 1 year	New number of heroin users
1990	200	0	0	200
1991	200	100	100	300
1992	300	–100	–100	200
1993	200	100	100	300
1994	300	300	300	600
1995	600	–100	–100	500
1996	500	0	0	500
1997	500	0	0	500
1998	500	0	0	500
1999	500	200	200	700

Table 2

The very last entry in the table (700) is the approximate number of high school seniors who were using heroin one year after 1999. Therefore, the year 2000 there were approximately 700 high school seniors using heroin.

3. The numbers calculated in Questions 1 and 2 are both estimates (or approximations) of the actual or "true" number of high school seniors who were using heroin in 2000. I would expect the number calculated in Question 2 (i.e. 700 heroin users) to be closer to the "true" number of heroin using high school seniors. This is because the size of the time interval used in Question 2 (a one year time interval) was shorter than the time interval used in Question 3 (a five year time interval). Generally, speaking, when you use a smaller time interval the accuracy of this method for estimating the values of the function improves.

4. The completed version of Table 3 is given below. Note that the numbers that you obtained might be slightly different than the numbers listed below because you rounded differently. So long as your numbers are close to the ones given below, you will be fine.

Year	Number of illicit drug users	Rate of change	Amount that number of users will change by in 2 years	New number of illicit drug users
1991	29,400	8970	17940	47340
1993	47340	–2691	–5382	41958
1995	41958	807.3	1614.6	43572.6
1997	43572.6	–242.19	–484.38	43088.22
1999	43088.22	72.657	145.314	43233.534
2001	43233.534	–21.7971	–43.5942	43189.9398
2003	43189.9398	6.53913	13.07826	43203.01806

Table 3

Therefore, in 2003 I would expect about 43189 or 43190 high school seniors to have tried illicit drugs.

5. Deciding whether the estimate of 43189 illicit drug using high school seniors for the year 2003 is quite tricky. Determining whether the number 43189 is an over- or an under-estimate is tricky because the rate of change oscillates between positive and negative values.

If the rate of change simply always increased, then you would be able to determine that the function $N(T)$ is a concave up function. You would then be justified in saying that the result of 43189 is certainly an under-estimate of the "true" number of high school seniors who will use illicit drugs in 2003. This is because our method of estimating function values under-estimates the values of a concave up function.

Likewise, if the rate of change simply always decreased then you would be able to determine that the function $N(T)$ is a concave down function. You would then be justified in saying that the result of 43189 is certainly an over-estimate of the "true" number of high school seniors who will use illicit drugs in 2003. This is because our method of estimating functions over-estimate the values of a concave down function.

These observations are summarized in the table given below.

Original Function	Derivative	Second Derivative	Estimated Function Values
Concave up	Increasing	Positive	Are lower than "true" function values
Concave down	Decreasing	Negative	Are higher than "true" function values

One way to investigate how the estimated and "true" values of $N(T)$ compare is to start with the equation for the derivative $N'(T)$ and differentiate. Differentiating the equation

$$N'(T) = -0.65 \cdot \left(N(T) - 43{,}200\right) = -0.65 \cdot N(T) + 28080,$$

gives:

$$N''(T) = -0.65 \cdot N'(T) = (-0.65)(-0.65)\left(N(T) - 43{,}200\right) = 0.4225 \cdot \left(N(T) - 43{,}200\right).$$

At the beginning (i.e. at $T = 0$) the number of illicit-drug using high school seniors was 29,400. If you substitute $T = 0$ and $N(T) = 29{,}400$ into the equation for $N''(T)$ you obtain:

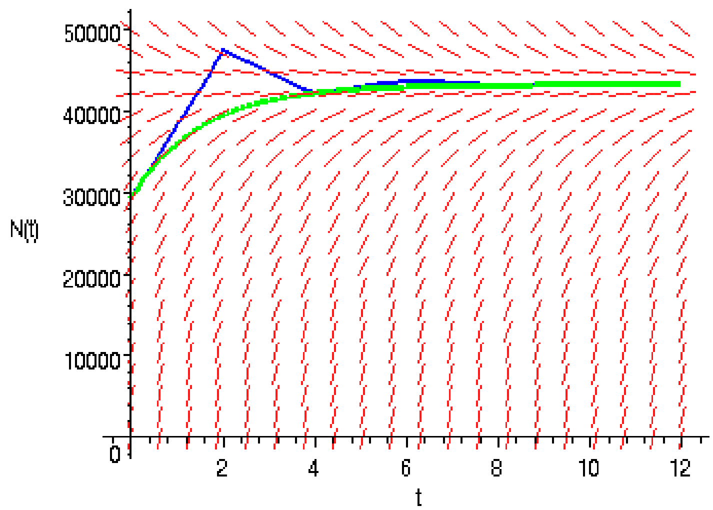
$$N''(0) = 0.4225 \cdot (29{,}400 - 43{,}200) = -5830.5.$$

From this you could conclude that when T is not too big, the second derivative $N''(T)$ is less than zero. Therefore, when T is not too big the function $N(T)$ is concave down. You would then expect the estimated values of the function to be higher than the "true" values of the function. So, on the basis of the sign of $N''(T)$, you would conclude that the number (about 43189 or 43190) obtained in Question 4 is an over-estimate of the true value of the function $N(T)$ for the year 2003.

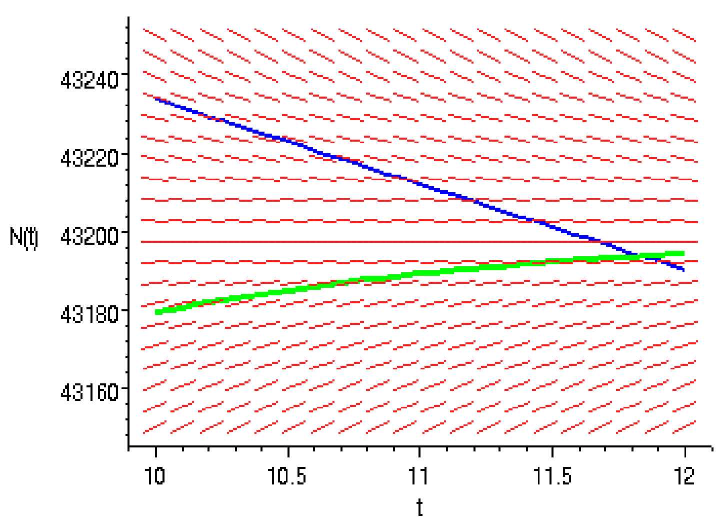
This analysis with the derivative is all that we were expecting you to do when working through this problem. For the vast majority of examples that you will encounter in Math Xb, this kind of analysis will do a very good job of determining whether estimated function values are higher or lower than the "true" function values. This kind of analysis will work perfectly in situations where the rate of change is always increasing or always decreasing.

However, as you can see from the table in Question 4, the rate of change is not always increasing, neither is it always decreasing. In this kind of situation, checking the sign of the second derivative at $T = 0$ is not always going to tell whether the estimated function values are higher or lower than the "true" function values. In situations where the rate of change sometimes goes up and sometimes goes down, you can use a slope field to sketch a plausible graph of the function. This will provide you with some idea of what the "true" values of the function are. You can then check to see whether the estimate you have calculated with the table seems to be higher or lower than the "true" value of the function, which you can get an idea of from your slope field and graph.

The slope field, the sketch of the function and the estimated function values from the table in Question 4 are shown below (the slope field is shown in red, the sketch of the function $N(T)$ is shown in green and the estimated values are shown in blue).



For 2003 ($T = 12$) it is not very easy to see if the green curve is higher or lower than the blue. To get a better idea of which is highest, you can zoom in on the region that surrounds $T = 12$.



From this graph, you can see that when T = 12, the blue line (representing the estimated function values from Question 4) is slightly lower than the green curve (representing the “true” function values). Therefore, despite what the second derivative $N''(T)$ suggested when we evaluated it for T = 0, the estimated value of the function is actually an under-estimate of the “true” value of $N(T)$ for T = 12.

Please note that we have presented this analysis so that you are aware that strange and unpredictable things can happen if the rate of change increases and decreases. In the vast majority of cases that you will encounter in Math Xb, the rate of change will either always increase or always decrease. If this is the case then you are perfectly safe using the second derivative at T = 0 to decide whether the estimated values of a function will be over-estimates or under-estimates of the “true” function values. However, as this example shows, it is definitely worth checking the behavior of the rate of change before drawing any definite conclusions about over- and under-estimates.