Homework Assignment 20: Solutions

1. Entering the data from Table 1 into a graphing calculator and performing a linear regression gave the equation:

$$L(T) = 2.593 \cdot T + 176.688.$$

A sketch of the graph of $y = L(T)$ together with the data from Table 1 is shown in Figure 1 below.

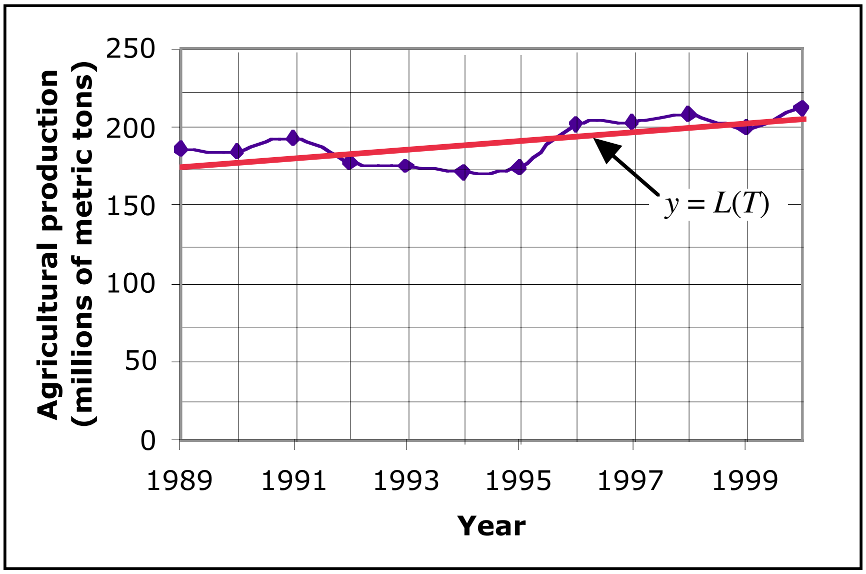


Figure 1: Sketch of $y = L(T)$ (the red line) and data from Table 1.

2. The completed version of Table 2 is given below.

Year	T	Agricultural Production	$L(T)$	Agricultural Production Minus $L(T)$
1989	0	186.2	176.688	9.512
1990	1	184.7	179.281	5.419
1991	2	194.0	181.874	12.126
1992	3	178.2	184.467	−6.267
1993	4	175.4	187.060	−11.660
1994	5	171.2	189.653	−18.453
1995	6	174.6	192.247	−17.647
1996	7	202.3	194.840	7.460
1997	8	203.6	197.433	6.167
1998	9	208.6	200.026	8.574
1999	10	199.9	202.619	−2.719
2000	11	212.7	205.212	7.488

Table 2.

The plot of (*Agricultural Production* – $L(T)$) versus T is shown in Figure 2 below.

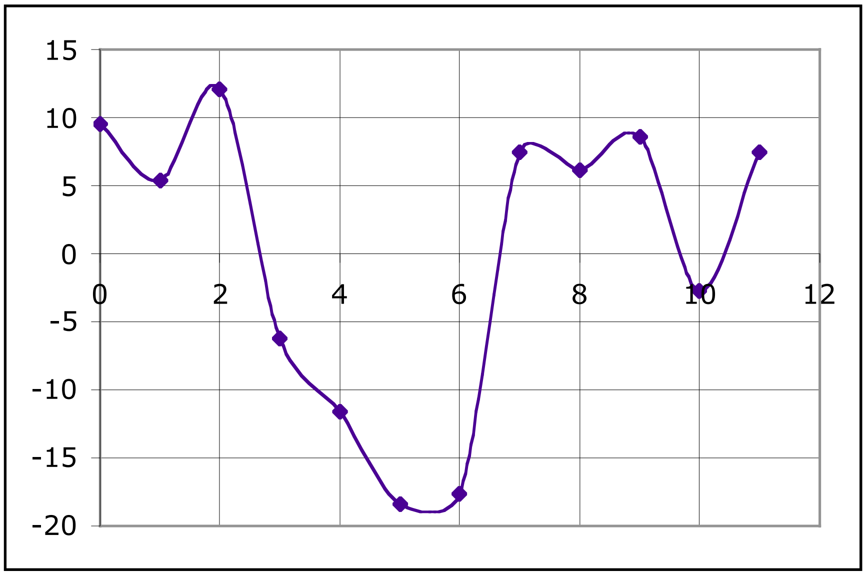


Figure 2: Plot of (*Agricultural Production* – $L(T)$) versus T.

3. The graph shown in Figure 2 is certainly not a perfect sine or cosine wave. However, its shape near $T = 0$ is reminiscent of the cosine curve shown in Figure 3 (below). Therefore, the functional form that will probably do the best job of representing the function shown in Figure 2 would be:

$$f(x) = A \cdot \cos\left(\tfrac{2\pi}{P} \cdot x\right) + M.$$

To find an equation to approximate the curve shown in Figure 2, we need to find values for:

- The **period**, *P*.
- The **midline**, *M*.
- The **amplitude**, *A*.

From Figure 2 and the numbers given in Table 2:

One repetition of the function (the distance from peak to peak) takes about 8 years. Therefore, the **period** of the periodic function is 8.

The midline is the average of the highest and lowest *y*-values. Therefore, the numerical value of the **midline** is:

$$Midline = \frac{12.126 + (-18.453)}{2} = -3.1635.$$

The amplitude is one half of the difference between the highest and lowest *y*-values. Therefore the numerical value of the **amplitude** is:

$$Amplitude = \frac{12.126 - (-18.453)}{2} = -15.2895.$$

Combining all of this information will give a trigonometric function, $f(T)$ that approximates the data shown in Figure 2. The equation for $f(T)$ is:

$$f(T) = 15.2895 \cdot \cos\left(\tfrac{2\pi}{8} \cdot T\right) - 3.1635.$$

The function for annual agricultural production of the United States, $A(T)$ will be the sum of the linear function $L(T)$ and the trigonometric function $f(T)$. The equation for $A(T)$ will therefore be:

$$A(T) = 173.5245 + 2.592 \cdot T + 15.2895 \cdot \cos\left(\tfrac{2\pi}{8} \cdot T\right).$$

The plot of the graph of $A(T)$ together with the original data is shown in Figure 3 (below).

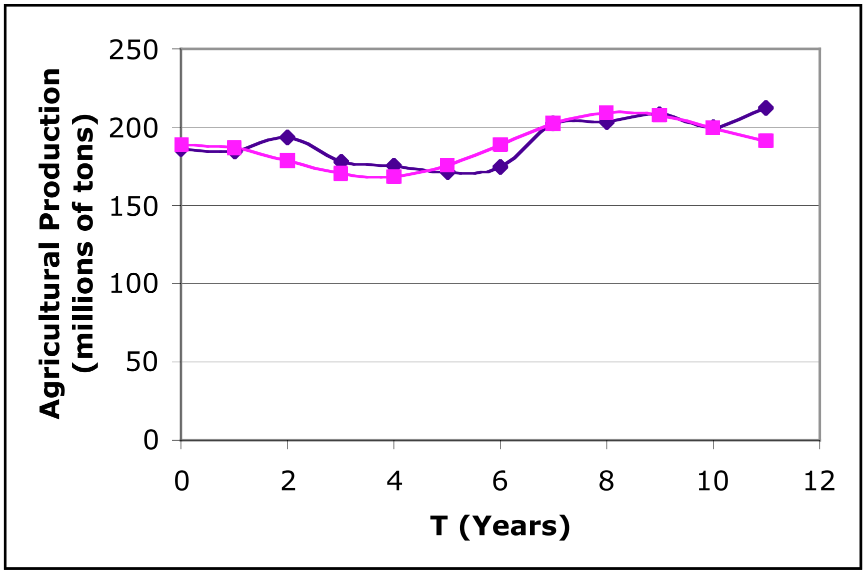


Figure 3: Plot of U.S. Agricultural Production (blue curve) and function $A(T)$ (purple curve), 1989-2000.

4. The equation for the first derivative of $A(T)$ is:

$$A'(T) = 2.592 - 15.2895 \cdot \sin\left(\tfrac{2\pi}{8} \cdot T\right) \cdot \tfrac{2\pi}{8} \approx 2.592 - 12.008 \cdot \sin\left(\tfrac{2\pi}{8} \cdot T\right).$$

The equation for the second derivative of $A(T)$ is:

$$A''(T) = -15.2895 \cdot \cos\left(\tfrac{2\pi}{8} \cdot T\right) \cdot \left(\tfrac{2\pi}{8}\right)^2 \approx -9.431 \cdot \cos\left(\tfrac{2\pi}{8} \cdot T\right).$$

To locate the critical points of a function, you set the first derivative equal to zero and then solve the resulting equation to find the value(s) of T. In this case, this will involve locating all solutions of the equation:

$$12.008 \cdot \sin\left(\tfrac{2\pi}{8} \cdot T\right) = 2.592.$$

Step 1: Use Algebra and the Inverse Trigonometry Functions to locate one solution.

$$12.008 \cdot \sin\left(\tfrac{2\pi}{8} \cdot T\right) = 2.592$$

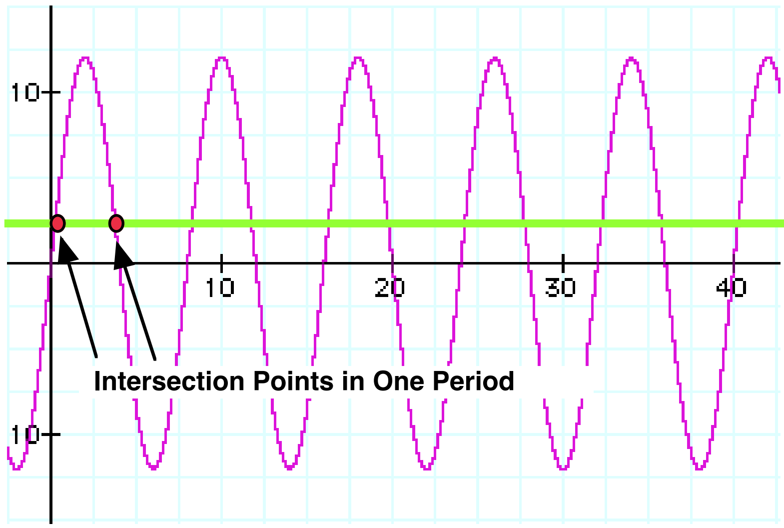
$$\sin\left(\tfrac{2\pi}{8} \cdot T\right) = 0.2158560959$$

$$\tfrac{2\pi}{8} \cdot T = \sin^{-1}(0.2158560959) = 0.2175685105$$

$$T = 0.2770168313.$$

Step 2: Use a graph of the function to locate all solutions within one period.

The graph given below shows the graph of $y = 12.008 \cdot \sin\left(\frac{2\pi}{8} \cdot T\right)$. The horizontal line is at a height of $y = 2.592$.



The first of the two intersection points is the one located in Step 1, T = 0.2770168313. The location of the second intersection point can be determined from the fact that the sine function is symmetric about its maximum and minimums. Therefore, the second intersection point is located at $T = 4 - 0.2770168313 = 3.722983169$.

Step 3: Create formulas giving all solutions by adding integer multiples of the period to the solutions that you found in Step 2.

The period of the function $y = 12.008 \cdot \sin\left(\frac{2\pi}{8} \cdot T\right)$ is equal to 8. The solutions of the trigonometric equation are given by:

$T = 0.2770168313 + 8 \cdot n$, where n is an integer.
$T = 3.722983169 + 8 \cdot n$, where n is an integer.

5. From Question 4, we know that the critical points of $A(T)$ are located at:

$T = 0.2770168313 + 8 \cdot n$, where n is an integer.
$T = 3.722983169 + 8 \cdot n$, where n is an integer.

What remains is to decide which of these points will be a local maximum and which will be a local minimum. Note that since sine and cosine are periodic functions (both with period equal to 2π)

$$\sin(\theta + 2\pi \cdot n) = \sin(\theta)$$
$$\cos(\theta + 2\pi \cdot n) = \cos(\theta)$$

so that:

$$\begin{aligned} A''(T + 8 \cdot n) &= -9.431 \cdot \cos\left(\tfrac{2\pi}{8} \cdot (T + 8 \cdot n)\right) \\ &= -9.431 \cdot \cos\left(\tfrac{2\pi}{8} \cdot T + 2\pi \cdot n\right) \\ &= -9.431 \cdot \cos\left(\tfrac{2\pi}{8} \cdot T\right) \\ &= A''(T) \end{aligned}$$

for integer values of n. Therefore, if we calculate the value of the second derivative of $A(T)$ at $T = 0.2770168313$, the second derivative will have exactly the same value at every point $T = 0.2770168313 + 8 \cdot n$, where n is an integer. Likewise, if we calculate the value of the second derivative of $A(T)$ at $T = 3.722983169$, the second derivative will have exactly the same value at each point $T = 3.722983169 + 8 \cdot n$, where n is an integer. Carrying out these calculations:

T	$A''(T)$	Interpretation
0.2770168313	–9.21	Local maximum
3.722983169	+9.21	Local minimum

Therefore the **local maximums** of agricultural production are located at the points:

$T = 0.2770168313 + 8 \cdot n$, where n is an integer.

The **local minimums** of agricultural production are located at the points:

$T = 3.722983169 + 8 \cdot n$, where n is an integer.