

Homework Assignment 21: Solutions

1. The total amount of agricultural produce grown in the United States from the beginning of 1989 to the end of 2000 will be the sum of all of the entries given in Table 1¹ (below).

Year	T	Agricultural production (millions of metric tons)
1989	0	186.2
1990	1	184.7
1991	2	194.0
1992	3	178.2
1993	4	175.4
1994	5	171.2
1995	6	174.6
1996	7	202.3
1997	8	203.6
1998	9	208.6
1999	10	199.9
2000	11	212.7

Table 1: Annual U.S. Agricultural Production, 1989-2000.

This is:

$$\begin{aligned}
 \text{Total amount of produce} &= 186.2 + 184.7 + 194.0 + 178.2 + 175.4 + \\
 & 171.2 + 174.6 + 202.3 + 203.6 + 208.6 + \\
 & 199.9 + 212.7 \\
 &= 2291.4 \text{ million metric tons.}
 \end{aligned}$$

2. Note that $T = 0$ will correspond to the beginning of the year 1989, and that $T = 12$ will correspond to the beginning of the year 2001, or to the very end of the year 2000.

The units the horizontal axis are **years** and the units of the vertical axis are **millions of metric tons per year**. Therefore the units of area under the graph given in the homework assignment will be:

$$\text{Units of Area} = \left(\frac{\text{Millions of metric tons}}{\text{Years}} \right) \cdot (\text{Years}) = \text{Millions of metric tons.}$$

The most technically correct interpretation of the area in question is this:

¹ Source: U.S. Department of Agriculture, Foreign Agricultural Service. *Grain: World Markets and Trade*. October, 2000. Available on-line from: http://www.fas.usda.gov/grain_arc.html

The area between the graph of $y=A(T)$ and the horizontal axis from $T = 0$ to $T = 12$ represents the net change in the total quantity of agricultural goods that has ever been produced in the United States, with this change occurring between the beginning of the year 1989 and the end of the year 2000, and the net change being expressed in units of millions of metric tons of agricultural produce.

A slightly less technically correct² (but far more intelligible) interpretation is:

The area between the graph of $y=A(T)$ and the horizontal axis from $T = 0$ to $T = 12$ represents the total amount of agricultural good produced in the United States between the beginning of 1989 and end of 2000, expressed in units of millions of metric tons.

3. Ultimately what we want to calculate in this problem is the numerical value of the integral:

$$\int_0^{12} A(T) \cdot dT.$$

To do this, we must first find an antiderivative for the function:

$$A(T) = 173.5245 + 2.592 \cdot T + 15.2895 \cdot \cos\left(\frac{2\pi}{8} \cdot T\right).$$

The antiderivatives of the first two terms of $A(T)$ are routine applications of the “short-cut” rules for finding antiderivatives. The antiderivative of the third term is obtained by the technique of u-substitution. Here, we will use u-substitution to calculate an equation for the indefinite integral (antiderivative):

$$\int 15.2895 \cdot \cos\left(\frac{2\pi}{8} \cdot T\right) \cdot dT.$$

Step 1: Choose u . $u = \frac{2\pi}{8} \cdot T$

Step 2: Calculate the derivative. $\frac{du}{dT} = \frac{2\pi}{8}$.

Step 3: Rearrange the derivative to make dT the subject. $dT = \frac{8}{2\pi} \cdot du$

Step 4: Rewrite the indefinite integral.

² The distinction that is blurred by the second interpretation is like the difference between the net change in your position and the total distance traveled. These will only be the same when your travels take you in one direction along a perfectly straight line. Otherwise the total distance traveled will usually be greater than the net change in position.

$$\int 15.2895 \cdot \cos\left(\frac{2\pi}{8} \cdot T\right) \cdot dT = (15.2895) \cdot \frac{8}{2\pi} \cdot \int \cos(u) \cdot du.$$

Step 5: Calculate the equation for the antiderivative using u as the variable.

$$(15.2895) \cdot \frac{8}{2\pi} \cdot \int \cos(u) \cdot du = (15.2895) \cdot \frac{8}{2\pi} \cdot \sin(u) + C.$$

Step 6: Rewrite the antiderivative in terms of x .

$$\int 15.2895 \cdot \cos\left(\frac{2\pi}{8} \cdot T\right) \cdot dT = (15.2895) \cdot \frac{8}{2\pi} \cdot \sin\left(\frac{2\pi}{8} \cdot T\right) + C$$

Using the results of this u -substitution together with the usual “short-cut” rules for antidifferentiation, the equation for the antiderivative of $A(T)$ is:

$$\int A(T) \cdot dT = 173.5245 \cdot T + \frac{2.592}{2} \cdot T^2 + (15.2895) \cdot \frac{8}{2\pi} \cdot \sin\left(\frac{2\pi}{8} \cdot T\right) + C.$$

To find the numerical value of the integral $\int_0^{12} A(T) \cdot dT$, we substitute $T = 12$ into this antiderivative, then substitute $T = 0$ into this antiderivative, and finally subtract these two values.

$$\begin{aligned} \int_0^{12} A(T) \cdot dT &= 2268.918 + C - (0 + C) \\ &= 2268.918 \text{ million metric tons of agricultural produce}^3. \end{aligned}$$

4. to find an equation for the antiderivative $\int \frac{\sin(\theta)}{\sqrt{1 - \sin^2(\theta)}} \cdot d\theta$ we can use the relationship:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

to simplify the equation. In particular, subtracting $\sin^2(\theta)$ from both sides of the relationship gives:

$$\cos^2(\theta) = 1 - \sin^2(\theta).$$

Therefore,

$$\frac{\sin(\theta)}{\sqrt{1 - \sin^2(\theta)}} = \frac{\sin(\theta)}{\sqrt{\cos^2(\theta)}} = \frac{\sin(\theta)}{\cos(\theta)},$$

³ Note that this result is very close to the figure calculated in Problem 1 (2291.4 million metric tons) for the total agricultural production of the United States from the beginning of 1989 to the end of 2000.

and so the antiderivative that we should find an equation for is really just:

$$\int \frac{\sin(\theta)}{\cos(\theta)} \cdot d\theta.$$

An equation for this antiderivative can be found using the technique of u-substitution.

Step 1: Select u . $u = \cos(\theta)$

Step 2: Calculate the derivative. $\frac{du}{d\theta} = -\sin(\theta)$

Step 3: Rearrange to make $d\theta$ the subject. $d\theta = \frac{du}{-\sin(\theta)}$

Step 4: Rewrite the indefinite integral.

$$\int \frac{\sin(\theta)}{\sqrt{1-\sin^2(\theta)}} \cdot d\theta = \int \frac{\sin(\theta)}{\cos(\theta)} \cdot d\theta = \int \frac{\sin(\theta)}{u} \cdot \frac{du}{-\sin(\theta)} = \int \frac{-1}{u} \cdot du$$

Step 5: Find an equation for the antiderivative regarding u as the variable.

$$\int \frac{-1}{u} \cdot du = -\ln(u) + C$$

Step 6: Rewrite the antiderivative in terms of θ .

$$\int \frac{\sin(\theta)}{\sqrt{1-\sin^2(\theta)}} \cdot d\theta = -\ln(\cos(\theta)) + C.$$

5. Before trying to find an equation for the antiderivative, we should try to simplify the algebraic expression that we have been asked to integrate as much as is mathematically possible. To achieve this, we were supplied with the formula for $\tan(\theta)$:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

The indefinite integral can then be written out, simplified and the antiderivative calculated:

$$\int \tan(\theta) \cdot \cos(\theta) \cdot d\theta = \int \frac{\sin(\theta)}{\cos(\theta)} \cdot \cos(\theta) \cdot d\theta = \int \sin(\theta) \cdot d\theta = -\cos(\theta) + C.$$