Homework Assignment 22: Solutions

The point of Questions 1-3 was to help you learn to recognize when to use the antiderivative formulas involving the inverse trigonometric functions. In each question you were presented with a pair integrals that seemed to be very similar. One of the integrals in the pair required the antidifferentiation formulas listed below, the other integral in the pair did not.

- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$
- $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1}(x) + C$
- $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$

1. The first integral to evaluate was: $\int \frac{2}{\sqrt{1-(2x)^2}} \cdot dx$. The u-substitution that would be the most desirable (as it would simplify this integral the most) would be $u = 1 - (2x)^2$. Differentiating $u = 1 - (2x)^2$ and rearranging to make dx the subject of the equation gives:

$$\frac{du}{dx} = -8x \qquad \text{and} \qquad dx = \frac{du}{-8x}.$$

Using $u = 1 - (2x)^2$ and the expression for dx to rewrite the integral gives:

$$\int \frac{2}{\sqrt{1-(2x)^2}} \cdot dx = \int \frac{2}{\sqrt{u}} \cdot \frac{du}{-8x}.$$

Unfortunately there is nothing for the x in the denominator of the integral to cancel with, so the choice of $u = 1 - (2x)^2$ must not be a "good" choice.

Going back to the drawing board and choosing $u = 2x$ gives $dx = \frac{1}{2} \cdot du$. Rewriting the integral with this choice of u gives:

$$\int \frac{2}{\sqrt{1-(2x)^2}} \cdot dx = \int \frac{2}{\sqrt{1-u^2}} \cdot \tfrac{1}{2} du = \int \frac{1}{\sqrt{1-u^2}} \cdot du = \sin^{-1}(u) + C = \sin^{-1}(2x) + C.$$

The second integral to evaluate was: $\int \frac{2x}{\sqrt{1-(2x)^2}} \cdot dx$. Again, the u-substitution that is the most attractive (from the point of view of simplifying the integral) is u

$= 1 - (2x)^2$. Differentiating and rearranging the derivative to make dx the subject gives $dx = \dfrac{du}{-8x}$. Using the expressions for u and dx to rewrite the integral gives:

$$\int \frac{2x}{\sqrt{1-(2x)^2}} \cdot dx = \int \frac{2x}{\sqrt{u}} \cdot \frac{du}{-8x} = \int \tfrac{-1}{4} u^{\frac{-1}{2}} \cdot du = \tfrac{-1}{2} u^{\frac{1}{2}} + C = \tfrac{-1}{2}\left(1-(2x)^2\right)^{\frac{1}{2}} + C$$

2. The first integral to be calculated was: $\int \dfrac{4x^3}{1+x^8} \cdot dx$. The u-substitution that would simplify this integral the most would be $u = 1 + x^8$. Differentiating this and rearranging the integral to make dx the subject of the expression gives:

$$\frac{du}{dx} = 8 \cdot x^7 \qquad \text{and} \qquad dx = \frac{du}{8 \cdot x^7}.$$

Using the expressions for u and dx to rewrite the integral gives:

$$\int \frac{4x^3}{1+x^8} \cdot dx = \int \frac{4x^3}{u} \cdot \frac{du}{8x^7} = \int \frac{1}{2} \cdot \frac{1}{u} \cdot \frac{1}{x^4} \cdot du.$$

Unfortunately, there is nothing left in the integral for the x^4 to cancel with. Therefore, the choice of $u = 1 + x^8$ was not a "good" choice for u. What would make a good choice for u? Generally speaking, when contemplating a u-substitution, you would like to see something in the integral that is closely related to the ***derivative*** of the algebraic expression that you want to use for u. If we were to try $u = x^4$, then the denominator in the integral would become $1 + u^2$ and the derivative of u would be $\frac{du}{dx} = 4x^3$, which appears in the numerator of the integral. This suggests that $u = x^4$ might be a "good" choice of u (although the only way to be sure is to try it). Rearranging the derivative to make dx the subject gives:

$$dx = \frac{du}{4x^3}.$$

Using $u = x^4$ and the expression for dx to rewrite the integral gives:

$$\int \frac{4x^3}{1+x^8} \cdot dx = \int \frac{4x^3}{1+u^2} \cdot \frac{du}{4x^3} = \int \frac{1}{1+u^2} \cdot du = \tan^{-1}(u) + C = \tan^{-1}\left(x^4\right) + C.$$

The second integral to evaluate was: $\int \dfrac{4x^7}{1+x^8} \cdot dx$. Again, the choice of u that has the greatest potential to simplify this expression is $u = 1 + x^8$. As before, differentiating u and rearranging the derivative to make dx the subject of the equation gives $dx = \dfrac{du}{8 \cdot x^7}$. Using u and dx to rewrite the integral gives:

$$\int \frac{4x^7}{1+x^8} \cdot dx = \int \frac{4x^7}{u} \cdot \frac{du}{8x^7} = \int \frac{1}{2} \cdot \frac{1}{u} \cdot du = \tfrac{1}{2} \cdot \ln(u) + C = \tfrac{1}{2} \cdot \ln\left(1 + x^8\right) + C.$$

3. The first integral to evaluate here was: $\int \frac{1}{\sqrt{1-x}} \cdot dx$. The only thing that is "inside" another algebraic structure is the (1 − x) that is inside the square root. An algebraic expression that is "inside" another is sometimes a good choice for u, so as a first attempt we can try $u = 1 - x$. Differentiating u with respect to x and rearranging the derivative to make dx the subject gives $dx = -du$. Rewriting the integral using u and dx gives:

$$\int \frac{1}{\sqrt{1-x}} \cdot dx = \int \frac{-1}{\sqrt{u}} \cdot du = \int -u^{\frac{-1}{2}} \cdot du = -2u^{\frac{1}{2}} + C = -2 \cdot (1-x)^{\frac{1}{2}} + C.$$

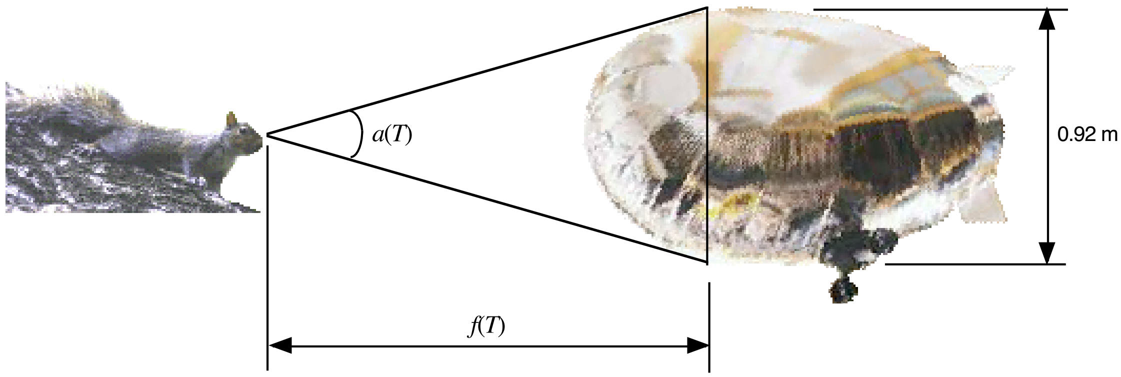
The second integral to evaluate was: $\int \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{1-x}} \cdot dx$. The u-substitution ($u = 1 - x$) that was very successful previously will not do such a good job here because there is an additional factor of $\frac{1}{\sqrt{x}}$ within the integral, and this additional factor has nothing to cancel with. As there are no particularly obvious choices for u we could just try the u suggested by the hint on the homework assignment: $u = \sqrt{x}$. Differentiating u with respect to x and rearranging the derivative to make dx the subject of the equation gives:

$$\frac{du}{dx} = \frac{1}{2 \cdot \sqrt{x}} \quad \text{and} \quad dx = 2 \cdot \sqrt{x} \cdot dx.$$

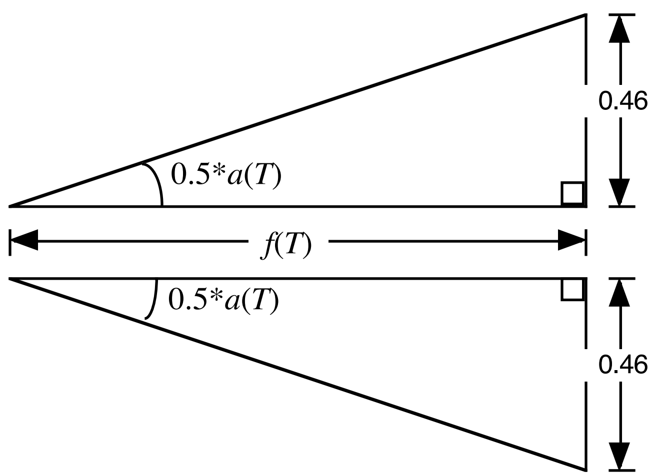
Noting that $x = \left(\sqrt{x}\right)^2$, we can use the expressions for u and dx to rewrite the integral. This gives:

$$\int \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{1-x}} \cdot dx = \int \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{1-u^2}} \cdot 2\sqrt{x} \cdot du = \int \frac{2}{\sqrt{1-u^2}} \cdot du = 2 \cdot \sin^{-1}(u) + C = 2 \cdot \sin^{-1}\left(\sqrt{x}\right) + C$$

On the homework assignment, you were presented with a diagram[1] (reproduced below) showing an object approaching a squirrel (*Sciurus carolinensis*). The object was moving and the squirrel was sitting still. In the diagram, the distance between the squirrel and the object is represented by the function $f(T)$. The apparent angle of the object (as seen by the squirrel) is represented by the function $a(T)$. T represents the time (in seconds) since the squirrel first spotted the approaching object.



4. To create a function for apparent angel, $a(T)$, we can break the triangle in the diagram down into two right-angle triangles as shown below.



Using the definition of tangent with either of these two triangles gives that:

$$\tan\left(\tfrac{1}{2}\cdot a(T)\right) = \frac{0.46}{f(T)}.$$

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[1] This figure was created using images from http://www.hammacher.com/ and http://www.animals-b-gone.com/

Using the inverse tangent function and algebraic manipulations to make $a(T)$ the subject of this equation gives:

$$a(T) = 2 \cdot \tan^{-1}\left(\frac{0.46}{f(T)}\right).$$

5. The expression for $a(T)$ given in the solution to Question 4 is a composite function, so the Chain Rule is the appropriate rule to use when differentiating $a(T)$. Doing this gives:

$$a'(T) = 2 \cdot \frac{1}{1+\left(\frac{0.46}{f(T)}\right)^2} \cdot \frac{-0.46 \cdot f'(T)}{\left[f(T)\right]^2}.$$

The numerical values provided in the homework assignment were:

- $f(T) = 3$.
- $f'(T) = -0.38$.

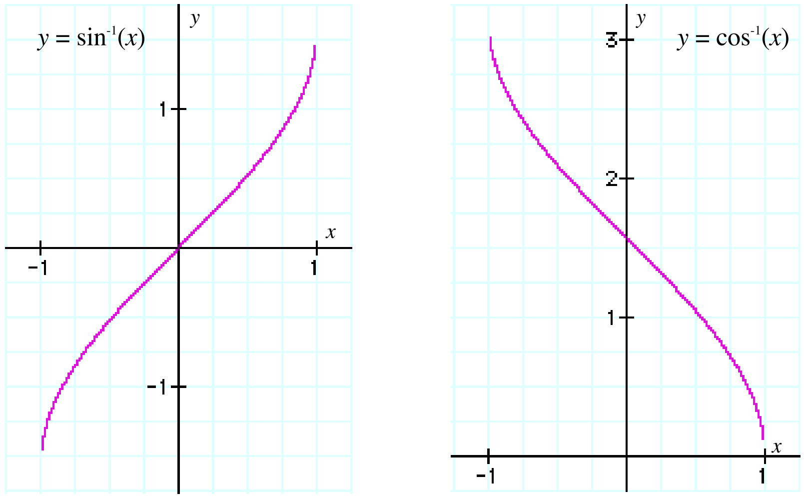
(Note that the derivative is **negative** because the distance between the squirrel and the object **decreases** as time passes.)

Substituting these numerical values into the equation for $a'(T)$ gives:

$$a'(T) = 0.03795 \text{ radians per second.}$$

Extra Credit.

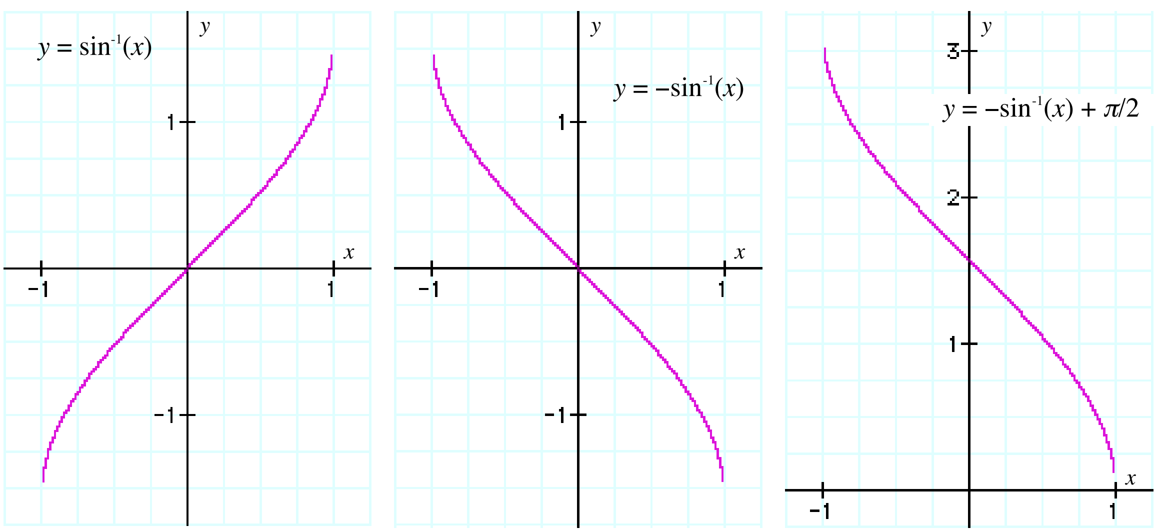
It is absolutely **NOT TRUE** that $\cos^{-1}(x) = -\sin^{-1}(x)$. This fact is clearly demonstrated by the graphs of the two functions in question (see below).



What **IS TRUE** is that:

$$\cos^{-1}(x) = -\sin^{-1}(x) + \tfrac{\pi}{2}.$$

One way to see that this relationship is true is to view the algebraic modifications of the inverse sine functions as geometric transformations of the graph of $y = \sin^{-1}(x)$. The two modifications are a reflection about the x-axis (represented by multiplying $\sin^{-1}(x)$ by -1 on the *outside*) followed by a vertical translation by $\frac{\pi}{2}$ (represented by adding $\frac{\pi}{2}$ to the *outside* of the function). The graphs given below show how these modifications of the basic inverse sine function transform the inverse sine function into the inverse cosine function.



How do these observations help to explain how it is possible for both of the antidifferentiation formulas

- $\int \frac{-1}{\sqrt{1-x^2}}\,dx = -\sin^{-1}(x) + C$
- $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1}(x) + C$

to be correct? The key lies in recognizing that the way that we normally write out antidifferentiation formulas is quite misleading. The most misleading part is the "eternal C," the generic constant term that is added to the end of an antiderivative. The "+C" is added to represent the possibility of the presence of a constant term in the antiderivative. However, by writing the same "+C" in both antidifferentiation formulas, it appears that not only is there a constant term present in each antiderivative, but that it is the *same* constant in both cases. This does not necessarily have to be the case. A much less confusing[2] way to write the two antiderivatives would be as

- $\int \frac{-1}{\sqrt{1-x^2}}\,dx = -\sin^{-1}(x) + C_1$
- $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1}(x) + C_2$

By writing a different constant (C_1 and C_2) in each antidifferentiation rule, it is easier to see that the two constants might have different values. If, in fact, we have that:

$$C_1 = C_2 + \tfrac{\pi}{2}$$

then using the relationship between the inverse sine and inverse cosine functions obtained above:

$$-\sin^{-1}(x) + C_1 = -\sin^{-1}(x) + \tfrac{\pi}{2} + C_2 = \cos^{-1}(x) + C_2.$$

Therefore, so long as the constants in the two antidifferentiation rules can be different, then both antidifferentiation rules can be correct despite the fact that $\cos^{-1}(x) \neq -\sin^{-1}(x)$.

[2] That will never catch on.