Homework Assignment 23: Solutions

1. The prototypical differential equation that usually serves as a good starting point for integrating the information on rates that you are given into a coherent equation is:

Rate at which Drug accumulates In body	=	*Rate at which enters body*	–	*Rate at which is eliminated from body*

In the problem, the following quantities were defined:

- T = the number of hours since the mouse was started on Botox®.
- $B(T)$ = the amount (in "units") of Botox® in the monkey's body.

The information that you were given about the rates involved was:

- The rate at which Botox® was supplied to the monkey was 0.0121 "units" per hour.
- The rate at which Botox® is eliminated from the body is proportional to the amount of Botox® in the monkey's body and the constant of proportionality is 0.00031.

Finally, you were also told that prior to the start of the experiment, the monkey did not have any Botox® in its body.

The initial condition should show that at the start of the experiment ($T = 0$) the monkey does not have any Botox® in its body. Symbolically, this could be expressed as:

$$B(0) = 0.$$

To determine the differential equation, note that the rate at which Botox® accumulates in the monkey's body will be represented by the symbol: $B'(T)$. Using this and the information given into the prototypical differential equation gives:

$$B'(T) = 0.0121 - 0.00031 \cdot B(T).$$

2. Using the differential equation:

$$B'(T) = 0.0121 - 0.00031 \cdot B(T)$$

and the initial condition $B(0) = 0$, it is possible to complete the table using Euler's method. The completed table is given below.

Time (hours)	Amount of Botox® ("units")	Rate of change	Change in amount of Botox® in next 30 minutes	New amount of Botox® ("units")
0	0	0.0121	0.00605	0.00605
0.5	0.00605	0.012098	0.006049	0.012099
1	0.012099	0.012096	0.006048	0.018471
1.5	0.018471	0.012094	0.006047	0.024518
2	0.024518	0.012092	0.006046	0.030564
2.5	0.030564	0.0120905	0.006045	0.036609

The very last entry (bottom right-hand corner of table) is the amount (in "units") of Botox® that have accumulated in the monkey's body after three hours. This is 0.036609 units of Botox®.

3. To determine whether or not the estimates of the values of $B(T)$ generated by Euler's method are over- or under-estimates of the "true values" of $B(T)$, you can examine the concavity of the function $B(T)$ by looking at the sign (+ or −) of the second derivative $B''(T)$. The table (see next page) shows the relationship between the second derivative, the concavity of $B(T)$ and whether Euler's method provides over- or under-estimates of the "true values" of $B(T)$.

To find the second derivative, you can differentiate the differential equation

$$B'(T) = 0.0121 - 0.00031 \cdot B(T)$$

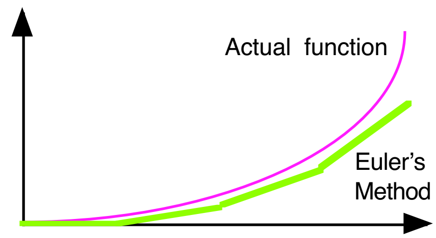
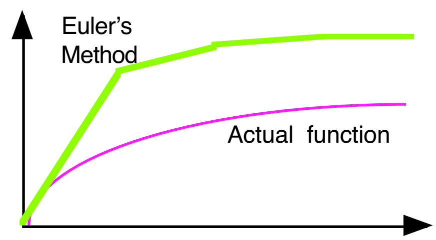
to give:

$$B''(T) = 0 - 0.00031 \cdot B'(T) = -0.00031 \cdot \left[0.0121 - 0.00031 \cdot B(T)\right]$$
$$= -0.000003751 + 0.0000000961 \cdot B(T)$$

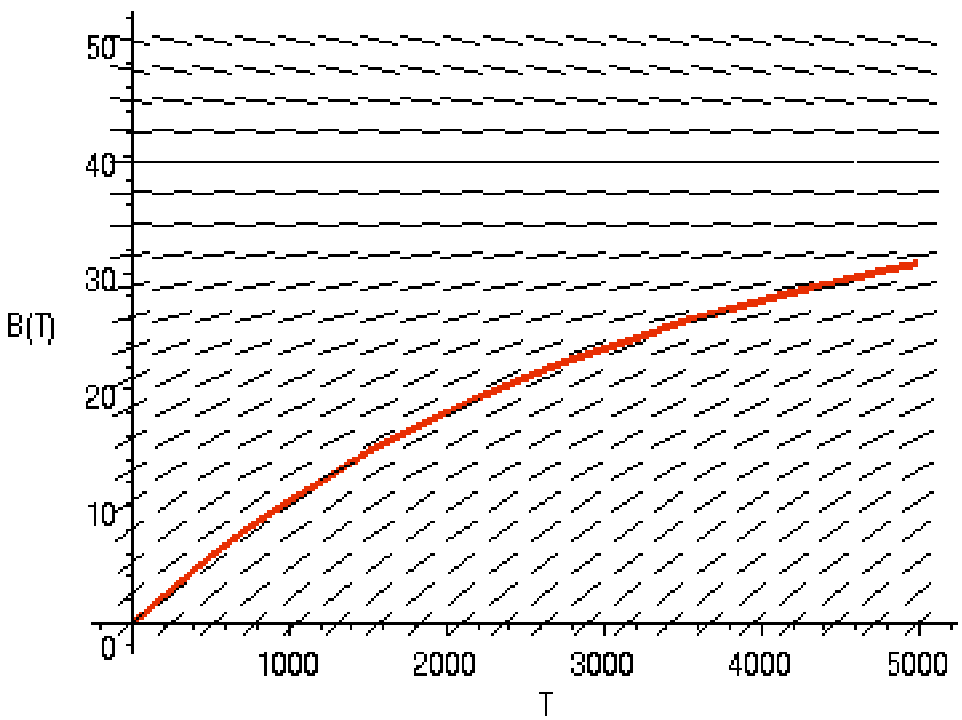
Substituting the initial value ($B(0) = 0$) into this equation gives

$$B''(0) = -0.000003751 < 0.$$

As the function $B(T)$ is concave down, the values obtained by using Euler's method will be **over-estimates** of the "true values" of the function $B(T)$.

Second derivative	Concavity of function	Picture	Numbers from Euler's method are:
Positive (+)	Concave up		Under-estimates of True function values
Negative (–)	Concave down		Over-estimates of True function values

4. The completed slope field (black) and graph of $B(T)$ versus T (red) is shown below. Note that the graph of $B(T)$ goes through the origin as required by the initial condition $B(T)$.



5. To demonstrate that the given function

$$B(T) = 39.032 - 39.032 \cdot e^{-0.00031 \cdot T}$$

is a solution of the differential equation and that this function satisfies the initial condition you can follow the steps given below.

i. Substitute the formula for the function into the **left hand side** of the differential equation.
ii. Substitute the formula for the function into the **right hand side** of the differential equation.
iii. If the results that you obtain from substituting the formula into the left and the right hand sides are the same, then the function is a solution of the differential equation.
iv. If you plug t=0 into the formula for the function and you obtain the initial value as a result, then the function also satisfies the initial value.

Step 1: Substituting the expression for $B(T)$ into the left hand side of the differential equation.

Left hand side $= B'(T)$

$= -0.00031 \cdot \left(-39.032 \cdot e^{-0.00031 \cdot T}\right)$

Step 2: Substituting the expression for $B(T)$ into the right hand side of the differential equation.

Right hand side $= 0.0121 - 0.00031 \cdot B(T)$

$= 0.0121 - 0.00031 \cdot \left(39.032 - 39.032 \cdot e^{-0.00031 \cdot T}\right)$

$= -0.00031 \cdot \left(-39.032 \cdot e^{-0.00031 \cdot T}\right)$

Step 3: Compare the expressions obtained in Steps 1 and 2.

The expressions obtained in Steps 1 and 2 are exactly the same. This indicates that the function defined by the formula

$$B(T) = 39.032 - 39.032 \cdot e^{-0.00031 \cdot T}$$

is a solution of the differential equation

$$B'(T) = 0.0121 - 0.00031 \cdot B(T).$$

Step 4: Check to see if the function also obeys the initial condition.

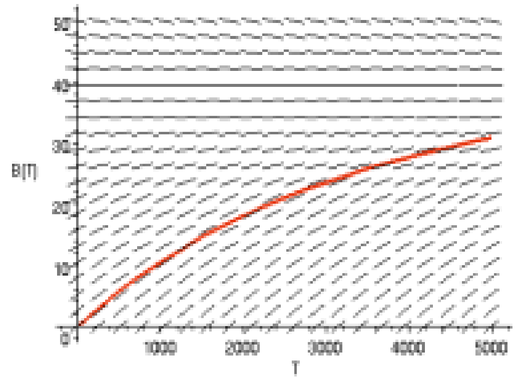
Substituting $T = 0$ into the function gives $B(0) = 39.032 - 39.032 = 0$.

Extra Credit.

The short answer is: No, this experiment is unlikely to prove lethal for an adult rhesus monkey.

To demonstrate that this is the case beyond a shadow of a doubt, we begin by noting that the dose of Botox® that will kill 50% of the monkeys that it is administered to (the LD50) is about 39 units per kg. For an adult rhesus monkey (body mass of about 7 kg) the LD50 will be 7×39 = 273 units of Botox®.

The appearance of the graph of $B(T)$ versus T from Question 4 (see also below) shows that the level of Botox® in the monkey's body is rising as time goes by, but also appears to be leveling off. The appearance of the slope field suggests that the amount of Botox® in the monkey's body will eventually level off and remain steady at about 40 units, well below the LD50 of 273 units.



To determine the exact level at which the amount of Botox® in the monkey's system will level off, you can do one of two things.

i. Use the differential equation $B'(T) = 0.0121 - 0.00031 \cdot B(T)$ to locate the equilibrium solution near $B(T) = 40$ units.

ii. Find the limiting value of $B(T)$ as $T \to +\infty$ using the equation for the function, $B(T) = 39.032 - 39.032 \cdot e^{-0.00031 \cdot T}$.

Following the first approach, setting $B'(T) = 0$ gives $B(T) = \frac{0.0121}{0.00031} \approx 39.032$ units of Botox®. Following the second approach, noting that as $T \to +\infty$, $39.032 \cdot e^{-0.00031 \cdot T} \to 0$, gives that as $T \to +\infty$, $B(T) \to 39.032$ units of Botox®.

Therefore, the maximum amount of Botox® that will accumulate in the monkey's body is about 39.032 units, which is well below the LD50 of 273 units.