## Homework Assignment 24: Solutions

In all of these *Separation of Variables* problems, the procedure is essentially the same. To help organize the work, we will present the solutions to the homework by following the steps outlined below. You may find these steps a useful guide for organizing your own work when performing the technique of *Separation of Variables*.

1. **Re-write the differential equation using the $\frac{dy}{dt}$ notation instead of the $y'(t)$ notation.** This will help you to carry out the algebraic manipulations in Step 3 more easily.
2. **It may also help to just write $y$ instead of $y(t)$ in the differential equation.**
3. **Separate the variables.** Rearrange the differential equation so that everything involving $y$ ends up on one side of the equation and everything involving $t$ ends up on the other side of the equation. You can put constants on whichever side of the equation you like.
4. **Add integral signs to both sides of the equation.**
5. **Integrate both sides of the equation *with respect to the appropriate variable*.** On the side of the equation where you have collected all of the $y$'s, the variable being integrated will be $y$. On the side of the equation where you have collected all of the $t$'s, the variable being integrated will be $t$.
6. **When you have finished integrating, rearrange the equation to make $y$ the subject (if necessary).**
7. **At this point, your formula for $y$ should have one unspecified constant of integration in it. Use the initial value to determine the numerical value of this constant.**

**1.** In this problem you were given:

**Differential Equation:** $y'(t) = \frac{1}{2} \cdot y(t)$.
**Initial value:** $y(0) = 5$.

Carrying out Steps 1-7 (outlined at the beginning of this set of solutions) to find a formula for $y$ as a function of $t$ using the technique of *Separation of Variables*:

**Step 1: Rewrite the differential equation using $\frac{dy}{dt}$ notation.**

$$\frac{dy}{dt} = \frac{1}{2} \cdot y(t)$$

**Step 2: Replace $y(t)$ with just plain $y$ in the differential equation.**

$$\frac{dy}{dt} = \frac{1}{2} \cdot y$$

**Step 3: Rearrange the differential equation so that all of the** $y$**'s are on one side of the equation and all of the** $t$**'s are on the other side of the equation.**

$$\frac{1}{y} \cdot dy = \frac{1}{2} \cdot dt$$

**Step 4: Add integral signs to both sides of what the differential equation has been rearranged into.**

$$\int \frac{1}{y} \cdot dy = \int \frac{1}{2} \cdot dt$$

**Step 5: Integrate both sides with respect to the appropriate variable.**

$$\ln(y) = \frac{1}{2} \cdot t + C$$

**Step 6: Rearrange the equation to make** $y$ **the subject (if necessary).**

You can remove the presence of the natural logarithm in the equation by exponentiating both sides with a base of $e$.

$$e^{\ln(y)} = e^{\frac{1}{2} \cdot t + C}$$

Using the law of exponents to simply both sides of the equation gives:

$$y = e^{C} \cdot e^{\frac{1}{2} \cdot t} = A \cdot e^{\frac{1}{2} \cdot t}, \text{ where } A = e^{C}.$$

**Step 7: Use the initial value to determine the numerical value of any constants in your formula for** $y$**.**

The initial condition is $y(0) = 5$. Substituting $t = 0$ into the formula from Step 6 and setting the whole thing equal to 5 gives: $A = 5$. Therefore, the explicit formula for $y$ as a function of $t$ is:

$$y = 5 \cdot e^{\frac{1}{2} \cdot t}.$$

**2.** In this problem you were given:

**Differential Equation:** $y'(t) = 4 \cdot y(t)$.
**Initial value:** $y(0) = 64$.

Carrying out Steps 1-7 (outlined at the beginning of this set of solutions) to find a formula for $y$ as a function of $t$ using the technique of *Separation of Variables*:

**Step 1: Rewrite the differential equation using $\frac{dy}{dt}$ notation.**

$$\frac{dy}{dt} = 4 \cdot y(t)$$

**Step 2: Replace *y*(*t*) with just plain *y* in the differential equation.**

$$\frac{dy}{dt} = 4 \cdot y$$

**Step 3: Rearrange the differential equation so that all of the *y*'s are on one side of the equation and all of the *t*'s are on the other side of the equation.**

$$\frac{1}{y} \cdot dy = 4 \cdot dt$$

**Step 4: Add integral signs to both sides of what the differential equation has been rearranged into.**

$$\int \frac{1}{y} \cdot dy = \int 4 \cdot dt$$

**Step 5: Integrate both sides with respect to the appropriate variable.**

$$\ln(y) = 4 \cdot t + C$$

**Step 6: Rearrange the equation to make *y* the subject (if necessary).**

You can remove the presence of the natural logarithm in the equation by exponentiating both sides with a base of $e$.

$$e^{\ln(y)} = e^{4 \cdot t + C}$$

Using the law of exponents to simply both sides of the equation gives:

$$y = e^C \cdot e^{4 \cdot t} = A \cdot e^{4 \cdot t}, \text{ where } A = e^C.$$

**Step 7: Use the initial value to determine the numerical value of any constants in your formula for *y*.**

The initial condition is $y(0) = 64$. Substituting $t = 0$ into the formula from Step 6 and setting the whole thing equal to 64 gives: $A = 64$. Therefore, the explicit formula for $y$ as a function of $t$ is:

$$y = 64 \cdot e^{4 \cdot t}.$$

**3.** In this problem you were given:

**Differential Equation:** $y'(t) = \frac{1}{2} \cdot [y(t) - 1]$.
**Initial value:** $y(0) = 9$.

Carrying out Steps 1-7 (outlined at the beginning of this set of solutions) to find a formula for *y* as a function of *t* using the technique of *Separation of Variables*:

**Step 1: Rewrite the differential equation using $\frac{dy}{dt}$ notation.**

$$\frac{dy}{dt} = \frac{1}{2} \cdot [y(t) - 1]$$

**Step 2: Replace *y*(*t*) with just plain *y* in the differential equation.**

$$\frac{dy}{dt} = \frac{1}{2} \cdot [y - 1]$$

**Step 3: Rearrange the differential equation so that all of the *y*'s are on one side of the equation and all of the *t*'s are on the other side of the equation.**

$$\frac{1}{y-1} \cdot dy = \frac{1}{2} \cdot dt$$

**Step 4: Add integral signs to both sides of what the differential equation has been rearranged into.**

$$\int \frac{1}{y-1} \cdot dy = \int \frac{1}{2} \cdot dt$$

**Step 5: Integrate both sides with respect to the appropriate variable.**

$$\ln(y - 1) = \frac{1}{2} \cdot t + C$$

**Step 6: Rearrange the equation to make *y* the subject (if necessary).**

You can remove the presence of the natural logarithm in the equation by exponentiating both sides with a base of *e*.

$$e^{\ln(y-1)} = e^{\frac{1}{2} \cdot t + C}$$

Using the law of exponents to simply both sides of the equation gives:

$$y - 1 = e^{C} \cdot e^{\frac{1}{2} \cdot t} = A \cdot e^{\frac{1}{2} \cdot t}, \text{ where } A = e^{C}.$$

Finally, adding one to both sides of the equation gives

$$y = 1 + A \cdot e^{\frac{1}{2} \cdot t}.$$

**Step 7: Use the initial value to determine the numerical value of any constants in your formula for *y*.**

The initial value is $y(0) = 9$. Substituting $t = 0$ and $y = 9$ into the formula from Step 6 gives:

$$9 = 1 + A$$

so that $A = 8$, and the final formula for $y$ as a function of $t$ is:

$$y = 1 + 8 \cdot e^{\frac{1}{2} \cdot t}.$$

**4.** In this problem you were given:

**Differential Equation:** $y'(t) = 3 \cdot [1 - y(t)]$.
**Initial value:** $y(0) = 10$.

Before carrying out the steps, we will rewrite the differential equation in a slightly different form.

$$y'(t) = -3 \cdot [y(t) - 1].$$

Although this rearrangement does not change the differential equation at all, this minor piece of algebra will make the antidifferentiation (Step 5) a little easier[1].

Now we will carry out Steps 1-7 (outlined at the beginning of this set of solutions) to find a formula for $y$ as a function of $t$ using the technique of *Separation of Variables*:

**Step 1: Rewrite the differential equation using $\frac{dy}{dt}$ notation.**

$$\frac{dy}{dt} = -3 \cdot [y(t) - 1]$$

**Step 2: Replace $y(t)$ with just plain $y$ in the differential equation.**

$$\frac{dy}{dt} = -3 \cdot [y - 1]$$

[1] Without the rearrangement of the differential equation, it would be necessary to do a $u$-substitution with $u = 1 - y$ in Step 5.

**Step 3: Rearrange the differential equation so that all of the** $y$'s **are on one side of the equation and all of the** $t$'s **are on the other side of the equation.**

$$\frac{1}{y-1} \cdot dy = -3 \cdot dt$$

**Step 4: Add integral signs to both sides of what the differential equation has been rearranged into.**

$$\int \frac{1}{y-1} \cdot dy = \int -3 \cdot dt$$

**Step 5: Integrate both sides with respect to the appropriate variable.**

$$\ln(y-1) = -3 \cdot t + C$$

**Step 6: Rearrange the equation to make** $y$ **the subject (if necessary).**

You can remove the presence of the natural logarithm in the equation by exponentiating both sides with a base of $e$.

$$e^{\ln(y-1)} = e^{-3 \cdot t + C}$$

Using the law of exponents to simply both sides of the equation gives:

$$y - 1 = e^{C} \cdot e^{-3 \cdot t} = A \cdot e^{-3 \cdot t}, \text{ where } A = e^{C}.$$

Finally, adding one to both sides of the equation gives

$$y = 1 + A \cdot e^{-3 \cdot t}.$$

**Step 7: Use the initial value to determine the numerical value of any constants in your formula for** $y$**.**

The initial value is $y(0) = 10$. Substituting $t = 0$ and $y = 10$ into the formula from Step 6 gives:

$$10 = 1 + A$$

so that $A = 9$, and the final formula for $y$ as a function of $t$ is:

$$y = 1 + 9 \cdot e^{-3 \cdot t}.$$

**5.** In this problem you were given:

**Differential Equation:** $y'(t) = 2 \cdot y(t) - 8$.
**Initial value:** $y(0) = 7$.

Before carrying out the steps, we will rewrite the differential equation in a slightly different form.

$$y'(t) = 2 \cdot [y(t) - 4].$$

Although this rearrangement does not change the differential equation at all, this minor piece of algebra will make the antidifferentiation (Step 5) a little easier[2].

Now we will carry out Steps 1-7 (outlined at the beginning of this set of solutions) to find a formula for $y$ as a function of $t$ using the technique of *Separation of Variables*:

**Step 1: Rewrite the differential equation using $\frac{dy}{dt}$ notation.**

$$\frac{dy}{dt} = 2 \cdot [y(t) - 4]$$

**Step 2: Replace $y(t)$ with just plain $y$ in the differential equation.**

$$\frac{dy}{dt} = 2 \cdot [y - 4]$$

**Step 3: Rearrange the differential equation so that all of the $y$'s are on one side of the equation and all of the $t$'s are on the other side of the equation.**

$$\frac{1}{y-4} \cdot dy = 2 \cdot dt$$

**Step 4: Add integral signs to both sides of what the differential equation has been rearranged into.**

$$\int \frac{1}{y-4} \cdot dy = \int 2 \cdot dt$$

**Step 5: Integrate both sides with respect to the appropriate variable.**

$$\ln(y-4) = 2 \cdot t + C$$

[2] Without the rearrangement of the differential equation, it would be necessary to do a $u$-substitution with $u = 2y - 8$ in Step 5.

**Step 6: Rearrange the equation to make *y* the subject (if necessary).**

You can remove the presence of the natural logarithm in the equation by exponentiating both sides with a base of *e*.

$$e^{\ln(y-4)} = e^{2\cdot t+C}$$

Using the law of exponents to simply both sides of the equation gives:

$$y-4 = e^{C}\cdot e^{2\cdot t} = A\cdot e^{2\cdot t}, \text{ where } A = e^{C}.$$

Finally, adding four to both sides of the equation gives

$$y = 4 + A\cdot e^{2\cdot t}.$$

**Step 7: Use the initial value to determine the numerical value of any constants in your formula for *y*.**

The initial value is $y(0) = 7$. Substituting $t = 0$ and $y = 7$ into the formula from Step 6 gives:

$$7 = 4 + A$$

so that $A = 3$, and the final formula for $y$ as a function of $t$ is:

$$y = 4 + 3\cdot e^{2\cdot t}.$$

[3]

[3] Image source: http://www.du.edu/