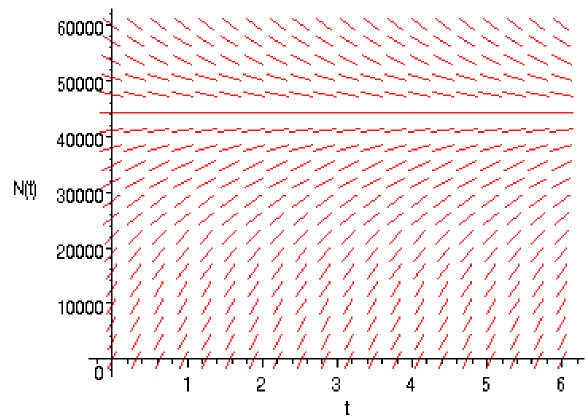
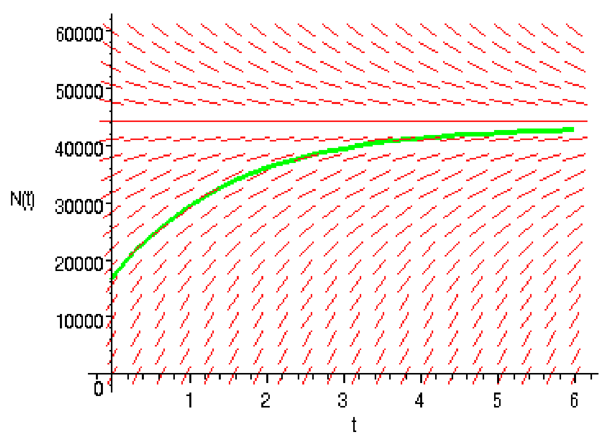
Homework Assignment 2: Solutions

1. The slope field for the equation: $N'(T) = -0.65 \cdot (N(T) - 43{,}200)$.
is given below. Clearly, the slope field given here was drawn with the aid of a specialized computer program[1]. So long as your slope field resembles this picture in its qualitative features, your answer will be perfectly acceptable.



2. The solution curve that passes through the point (1, 29400) is shown (in green) on the slope field given below. Again, this graph was produced with the aid of a computer program[2]. So long as your graph is consistent with the qualitative features of the picture shown below, your answer will be perfectly acceptable.



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[1] The program used for this picture was the computer algebra system Maple 6. If you are familiar with Maple 6, the specific commands used to produce the picture were:

```
[> with(DEtools):
[> dfieldplot(diff(N(t),t)=-0.65*(N(t)-43200),N(t),t=0..6,N=0..60000,
   arrows=line);
```

[2] The Maple 6 program was also used to produce this plot. The commands for this plot were:

```
[> with(DEtools):
[> DEplot(diff(N(t),t)=-0.65*(N(t)-43200),N(t),t=0..6,N=0..60000,
    [[N(1)=29400]],linecolor=green,arrows=line);
```

3. The equilibrium solution(s) of the equation:

$$N'(T) = -0.65 \cdot \left(N(T) - 43,200\right).$$

are the places where the derivative $N'(T)$ is equal to zero. Setting $N'(T) = 0$ gives:

$$-0.65 \cdot \left(N(T) - 43200\right) = 0$$

$$N(T) - 43200 = 0$$

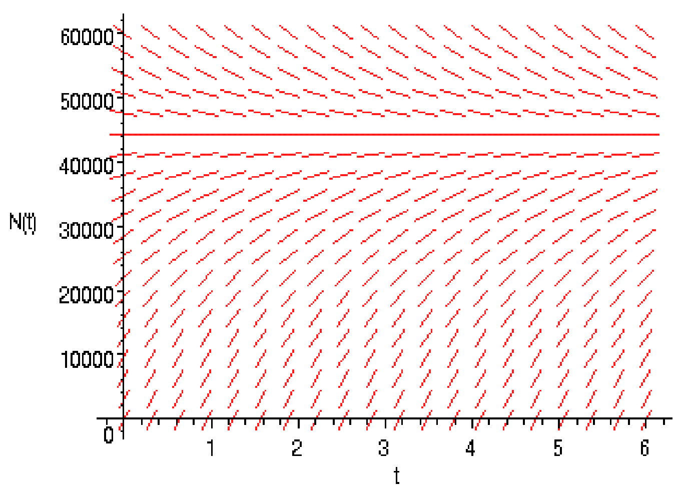
$$N(T) = 43200.$$

Therefore there is only one equilibrium solution. Graphically, it is a horizontal line at a height of 43200.

To classify an equilibrium solution, it is usually best to consult the slope field and read the slope field from left to right.

- If the little line segments appear to be attracted towards the equilibrium from above and from below then the equilibrium solution is a **stable equilibrium**.
- If the little line segments appear to be repelled away from the equilibrium (both above and below) then the equilibrium solution is an **unstable equilibrium**.
- If the little line segments are attracted on one side of the equilibrium but repelled on the other side, then the equilibrium solution is a **semi-stable equilibrium**.

The slope field for the equation: $N'(T) = -0.65 \cdot \left(N(T) - 43,200\right)$ is given below. Near the height of $N(T)$ = 43200, the little line segments appear to be attracted towards the equilibrium. Therefore, $N(T)$ = 43200 is a **stable equilibrium**.



4. The answer is: **(c)**.

How could you discern that the answer should be (c)? Here is one approach:

• If you have a look at the slope field, you can discern some places where there are probably equilibrium solutions. The two places that stand out are the horizontal lines:

$$y(t) = 0 \qquad \text{and} \qquad y(t) = 2.$$

Therefore, the equation that you select should have equilibrium solutions at both of these locations. If you check the equations that you are given, you will be able to see that all of these equations have $y'(t) = 0$ on the horizontal lines $y(t) = 0$ and $y(t) = 2$. That doesn't help much in this particular problem, but it might help to eliminate some possibilities in another problem.

• Next, if you examining the slope field, you can discern the nature of each of these equilibrium solutions. They are:

- $y(t) = 0$ is an unstable equilibrium
- $y(t) = 2$ is a semi-stable equilibrium.

In an equation like:

$$y'(t) = y(t)\cdot[y(t)-2]^2$$

factors that are raised to an odd power (1, 3, 5, etc.) usually produce stable or unstable equilibrium solutions in the slope field. Factors that are raised to an even power usually produce semi-stable equilibrium solutions in the slope field. So you would expect the slope field of an equation like

$$y'(t) = y(t)\cdot[y(t)-2]^2$$

to show either a stable or an unstable equilibrium at $y(t) = 0$ and a semi-stable equilibrium at $y(t) = 2$. This observations narrows the possible choices down to the equations **(a)** and **(d)**.

• Finally, if you have a look at the slope field, you can see that when $y(t) = 3$, the little line segments are pointing upwards. This indicates that when $y(t) = 3$, $y'(t) > 0$. If we go ahead and substitute $y(t) = 3$ into equations **(a)** and **(d)** we get:

Equation (a): $\quad y'(t) = y(t)\cdot[y(t)-2] = 3\cdot[3-2] = 3 > 0$.

Equation (d): $\quad y'(t) = -y(t)\cdot[y(t)-2] = -3\cdot[3-2] = -3 < 0$.

Therefore, only equation **(a)** is completely consistent with the appearance of the slope field that was given in the homework assignment.

5. The answer is: **(I)**.

To see how you could possibly arrive at this conclusion, you can go through an analysis similar to that presented in the solution to Question 4.

- Firstly, if you factor the equation then you can obtain:

$$y'(t) = y(t)\cdot\left[y(t)^2 - 1\right] = y(t)\cdot\left[y(t)+1\right]\cdot\left[y(t)-1\right].$$

The derivative $y'(t)$ will be equal to zero when $y(t) = 0$, $y(t) = -1$ and when $y(t) = 1$. This suggests that there will be three equilibrium solutions, and that they will resemble horizontal lines located at heights 0, −1 and 1. This observation immediately eliminates slope fields **(II)** and **(III)** from contention.

- Next, from the fact that the power of each factor in the equation:

$$y'(t) = y(t)\cdot\left[y(t)^2 - 1\right] = y(t)\cdot\left[y(t)+1\right]\cdot\left[y(t)-1\right]$$

is equal to 1, you would expect each of the equilibrium solutions to be a stable or an unstable equilibrium. Inspecting the slope fields **(I)** and **(IV)** reveals that all of the equilibrium solutions on these slope fields are either stable or unstable, so this point does not help you to decide between slope fields **(I)** and **(IV)**.

- The final test is to select a value for $y(t)$ and substitute this into the equation for $y'(t)$. Let's try $y(t) = 2$. Then:

$$y'(t) = y(t)\cdot\left[y(t)+1\right]\cdot\left[y(t)-1\right] = 2\cdot[2+1]\cdot[2-1] = 6 > 0.$$

As the derivative is positive for $y(t) = 2$, you would expect the little line segments on the slope field to point upwards when $y(t) = 2$. Slope field **(I)** does this, whereas slope field **(IV)** does not.