Homework Assignment 3: Solutions

1. The differential equation in this problem is:

$$N'(T) = -0.65 \cdot \left(N(T) - 43,200\right).$$

The best way to check to see whether a given function is a solution of a differential equation is to go through the following three steps:

- Substitute the function that you are given into the **left side** of the differential equation. In the case of the differential equation shown above, the left side of the differential equation is $N'(T)$.
- Substitute the function that you are given into the **right side** of the differential equation. In the case of the differential equation shown above, the left side of the differential equation is $-0.65 \cdot \left(N(t) - 43,200\right)$.
- Check to see if what you have obtained is the same quantity each time.

The functions that were offered for your consideration were:

(I) $N(T) = 43200 - 26434.46 \cdot e^{-0.65T}$.

(II) $N(T) = 29400 \cdot \sqrt{T}$.

(III) $N(T) = 43200 - 13800 \cdot e^{-0.65T}$.

Checking Function (I): $N(T) = 43200 - 26434.46 \cdot e^{-0.65T}$.

- **Left side:** $N'(T) = -0.65 \cdot \left(-26434.46 \cdot e^{-0.65T}\right)$
- **Right side:** Substituting $N(T) = 43200 - 26434.46 \cdot e^{-0.65T}$ gives

$$-0.65 \cdot \left(N(t) - 43,200\right) = -0.65 \cdot \left(43200 - 26434.46 \cdot e^{-0.65T} - 43200\right)$$

$$= -0.65 \cdot \left(-26434.46 \cdot e^{-0.65T}\right)$$

- **Comparison:** The left hand side and the right hand side are equal.
- **Conclusion:** $N(T) = 43200 - 26434.46 \cdot e^{-0.65T}$ is a solution of the differential equation.

Checking Function (II): $N(T) = 29400 \cdot \sqrt{T}$.

- **Left side:** $N'(T) = 29400 \cdot \frac{1}{2} T^{-1/2} = 14700 \cdot \frac{1}{\sqrt{T}}$

- **Right side:** Substituting $N(T) = 29400 \cdot \sqrt{T}$ gives

$-0.65 \cdot (N(t) - 43,200)$ = $-0.65 \cdot \left(29400 \cdot \sqrt{T} - 43200\right)$

- **Comparison:** The left hand side and the right hand side are not equal.

- **Conclusion:** $N(T) = 29400 \cdot \sqrt{T}$ is not a solution of the differential equation.

Checking Function (III): $N(T) = 43200 - 13800 \cdot e^{-0.65T}$.

- **Left side:** $N'(T) = -0.65 \cdot \left(-13800 \cdot e^{-0.65T}\right)$

- **Right side:** Substituting $N(T) = 43200 - 13800 \cdot e^{-0.65T}$ gives

$-0.65 \cdot (N(t) - 43,200)$ = $-0.65 \cdot \left(43200 - 13800 \cdot e^{-0.65T} - 43200\right)$

= $-0.65 \cdot \left(-13800 \cdot e^{-0.65T}\right)$

- **Comparison:** The left hand side and the right hand side are equal.

- **Conclusion:** $N(T) = 43200 - 13800 \cdot e^{-0.65T}$ is a solution of the differential equation.

2. To check to see if any of these functions are consistent with the information about the number of illicit drug using high school seniors in 1991, you can evaluate the functions for $T = 1$, and see whether you get a result of 29,400 or not. For the three functions given in Question 1, the results are summarized in the table below.

Function	Equation	N(1)	Consistent with information?
(I)	$N(T) = 43200 - 26434.46 \cdot e^{-0.65T}$	29400	Yes
(II)	$N(T) = 29400 \cdot \sqrt{T}$	29400	Yes
(III)	$N(T) = 43200 - 13800 \cdot e^{-0.65T}$	35995.77	No

3. In this problem the differential equation is:

$$T'(t) = -0.0033 \cdot [T(t) - 20],$$

the initial condition is

$$T(0) = 100,$$

and the function that you are given is

$$T(t) = 20 + 80 \cdot e^{-0.0033t}.$$

Checking that $T(t) = 20 + 80 \cdot e^{-0.0033t}$ solves the differential equation:

- **Left side:** $T'(t) = -0.0033 \cdot [80 \cdot e^{-0.0033t}]$
- **Right side:** Substituting $T(t) = 20 + 80 \cdot e^{-0.0033t}$ gives

$$-0.0033 \cdot [T(t) - 20] = -0.0033 \cdot [20 + 80 \cdot e^{-0.0033t} - 20]$$

$$= -0.0033 \cdot [80 \cdot e^{-0.0033t}]$$

- **Comparison:** The left side and the right side are equal.
- **Conclusion:** The given function satisfies the differential equation.

Checking that $T(t) = 20 + 80 \cdot e^{-0.0033t}$ is compatible with the initial condition:

- Substituting $t = 0$ into the given function gives: $T(0) = 20 + 80 = 100$.

4. To locate the equilibrium solutions of a differential equation such as

$$T'(t) = -0.0033 \cdot [T(t) - 20],$$

you can set $T'(t) = 0$ and then solve the resulting equation for $T(t)$. This yields only one equilibrium solution, which is:

$$T(t) = 20.$$

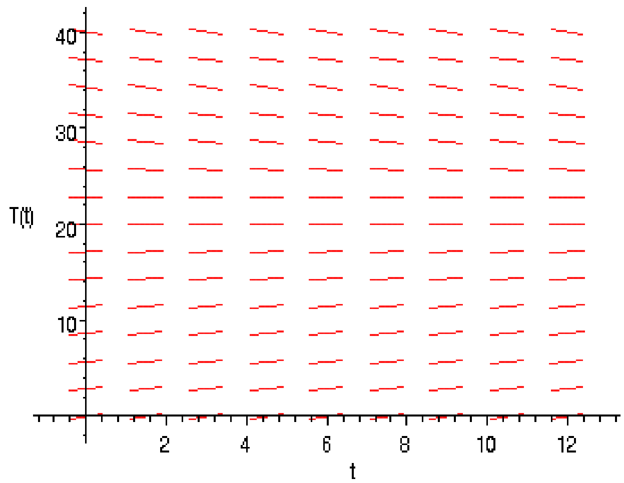
To classify an equilibrium solution as stable, unstable or semi-stable, it is usually best to draw the slope field based (using the differential equation) and see what the curves near the equilibrium point are doing. Remember that:

- **Stable** equilibrium solutions **attract** curves from above and below as $t \to \infty$.

• **Unstable** equilibrium solutions **repel** curves from above and below as $t \to \infty$.

• **Semi-stable** equilibrium solutions **attract** solutions from one side **and repel** solutions from the other side as $t \to \infty$.

The slope field for the differential equation $T'(t) = -0.0033 \cdot [T(t) - 20]$ is shown below. As you can see from the arrangement of the little line segments near the horizontal line *T*(*t*) = 20, this equilibrium solution attracts curves towards it, so *T*(*t*) = 20 is a stable equilibrium solution.



The interpretation of *T*(*t*) = 20 in terms of temperatures is that this is the temperature that the water in the Hammacher-Schlemmer cup will eventually approach as it cools down over a long period of time. In other words, the "room temperature" of the room in which the Hammacher-Schlemmer cup is placed is 20ºC.

One way to see that the existence of a stable equilibrium at *T*(*t*) = 20 is a result that you might expect is to reason like this: if you put a hot object into a room that is maintained at 20ºC then you would expect the object to cool (i.e. its temperature to decrease) until it had cooled down to approximately room temperature. Graphically, that means that if you started out with an initial value of temperature higher than 20ºC, you would expect the temperature curve to get closer and closer to a height of 20 - in other words, the temperature curve would be attracted towards the horizontal line at height 20.

On the other hand, if you put a cold object into a room where the temperature was maintained at a steady 20ºC, you would expect the object to warm up - that is, for its temperature to rise until the object was at about the same temperature as the room. Graphically, this would mean that even if the temperature curve started with an initial value of less than 20, you would expect the temperature curve to still be attracted towards the horizontal line at height 20.

5. To see what the pattern is, suppose that *B*, *C* and *k* are all constants. Suppose that the differential equation that you are given is:

$$P'(t) = k \cdot [P(t) - B]$$

and that the initial condition that you are given is:

$$P(0) = C.$$

Then the equation for the function $P(t)$ that is a solution of the differential equation, and that also satisfies the initial condition is:

$$P(t) = B + (C - B) \cdot e^{k \cdot t}.$$

Applying this pattern to the particular differential equation and initial condition given on the homework

$$P'(t) = 0.01 \cdot [P(t) - 400]$$

$$P(0) = 100.$$

suggests that the equation for the function $P(t)$ should be:

$$P(t) = 400 - 300 \cdot e^{0.01 \cdot t}.$$

Checking that $P(t) = 400 - 300 \cdot e^{0.01 \cdot t}$ **solves the differential equation:**

- **Left side:** $P'(t) = 0.01 \cdot [-300 \cdot e^{0.01 \cdot t}]$
- **Right side:** Substituting $P(t) = 400 - 300 \cdot e^{0.01 \cdot t}$ gives

$$\begin{aligned} 0.01 \cdot [P(t) - 400] &= 0.01 \cdot [400 - 300 \cdot e^{0.01 \cdot t} - 400] \\ &= 0.01 \cdot [-300 \cdot e^{0.01 \cdot t}] \end{aligned}$$

- **Comparison:** The left side and the right side are equal.
- **Conclusion:** The given function satisfies the differential equation.

Checking that $P(t) = 400 - 300 \cdot e^{0.01 \cdot t}$ **is compatible with the initial condition:**

- Substituting $t = 0$ into the given function gives: $P(0) = 400 - 300 = 100$.