Homework Assignment 4: Due at the beginning of class 2/13/02

In the first week of class, you encountered many *differential equations*. You encountered a differential equation describing the number of illicit-drug using high school seniors on homework assignments 1, 2, and 3. On homework 3, you also encountered Newton's Law of Cooling expressed as a differential equation. You may also have encountered differential equations describing the number of wrestlers infected with *Herpes gladiatorum* and the amount of alcohol in a person's stomach.

You may have wondered where these differential equations come from. The idea of Question 1 is to guide you through the formulation of a differential equation based on a verbal description of a situation.

1. Your objective in this question is to create a differential equation and initial condition that describe the quantity (milligrams, mg) of a drug in a patient's body. At the end of the question, you will substitute a function known to describe this situation into your differential equation and initial condition as a validation of your work.

Figure 1: Infusion bottles of Naropin ®.

Ropivacaine is a local anesthetic used for controlling pain during surgery and childbirth, and is also administered after surgery for the management of post-operative pain[1]. Ropivacaine is manufactured by AstraZeneca corporation and sold under the brand name *Naropin*®[2]. Figure 1[3] shows the infusion bottles that large quantities of Naropin® are usually supplied in. Ropivacaine obeys the linear law of pharmacokinetics. This means that the rate at which ropivacaine is eliminated from the body is proportional to the quantity of ropivacaine in the body. For ropivacaine (and hence for Naropin®), the constant of proportionality[4] is 0.385.

For the management of post-operative pain, Naropin® is often administered to a patient via a device called a *pain pump* (see Figure 2[5]). A pain pump delivers a drug to the patient at a measured and very steady rate. According to a study reported in the *British Journal of*

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[1] Source: http://www.astrazeneca.com/Patients/products/product_Ropivacaine.html
[2] "Naropin" is a registered trademark of AstraZeneca International.
[3] Image source: http://www.naropin-us.com/postop.asp
[4] Source: http://healthanswers.telstra.com/drugdata/appco/00097100.asp Please note that the constant of proportionality given here is only valid for individuals over the age of 12. For children the constant of proportionality is considerably lower than 0.385.
[5] Image source: http://www.pain-pump.com/

Anaesthesiology[6], when Naropin® is supplied at a rate of 28 mg per hour, 90% of patients reported their pain management to be "good" or "excellent."

Figure 2: A pain pump for continuous dosage of a pain-relieving drug such as a local anesthetic or morphine. The patient normally has some kind of control to increase the flow of the medication when pain becomes acute.

Imagine an adult patient who has just undergone major surgery. Assume that during surgery, a drug other than ropivaraine was employed as an anesthetic. Immediately after the surgery, however, the patient is placed on a pain pump supplying ropivaraine in the form of Naropin®.

Let T represent the number of hours that a patient has been on the pain pump and $R(T)$ the quantity (in milligrams, mg) of Naropin® in the patient's body. Your job is to create a differential equation and initial condition that describe this situation.

A potentially useful starting point for formulating this differential equation is the prototype:

rate of change of Naropin® in patient's body	=	rate that Naropin® enters body	−	rate that Naropin® leaves body

The formula for the function $R(T)$ is given below.

$$R(T) = \frac{7200}{99} - \frac{7200}{99} \cdot e^{-0.385 \cdot T}.$$

As the final part of your answer to Question 1, verify that this formula for $R(T)$ is consistent with the differential equation and initial condition that you have created.

[6] Source: G. Turner, D. Blake and M. Buckland. (1996) "Continuous extradural infusion of ropivacaine for prevention of postoperative pain after major orthopaedic surgery." *British Journal of Anaesthesiology,* **76**: 606-610.

Figure 3: Insulated commuter cup tested as best by the Hamaacher-Schlemmer institute.

The Hammacher-Schlemmer Institute carries out tests of consumer items to select the best versions of various consumer goods. One of the kinds of products recently tested was insulated beverage mugs. The very best mug (according to the Hammacher-Schlemmer tests) is shown in Figure 3[7]. (This is the same mug that kept showing up in Math Xa.)

The rate at which the temperature of a hot beverage changes is given by the equation known as Newton's Law of Cooling. This law states that:

$$T'(t) = k \cdot \left[T(t) - T_{room} \right]$$

where $T(t)$ is the temperature of the beverage after t minutes, T_{room} represents the temperature of the room, and k is a number determined by the insulating properties of the beverage container. Suppose that the room temperature is 20°C, and that the liquid in the Hammacher-Schlemmer cup starts out at 100°C. (That is, $T(0) = 100$.) The constant in Newton's Law of Cooling has not been specified, so just use the symbol k to represent it.

Normally, when you have seen the Hammacher-Schlemmer cup, you have used Euler's Method, together with a definite numerical value of k, to approximate the value of the temperature of the beverage in the cup. The idea of Questions 2-5 is to do the same sort of calculations, but with the intention of coming up with a *formula* for the estimated temperature of the liquid in the Hammacher-Schlemmer cup, rather than numerical values. This calculation is going to lead you into the topic of **geometric series**, which you will be working with in lab and section this week.

2. Show that when $t = 1$, the temperature of the beverage is:

$$T(1) = (1 + k)\cdot 100 - 20\cdot k.$$

3. Show that when $t = 2$, the temperature of the beverage is:

$$T(2) = (1 + k)^2\cdot 100 - 20\cdot k\cdot(1 + k) - 20\cdot k.$$

4. Show that when $t = 3$, the temperature of the beverage is:

$$T(3) = (1 + k)^3\cdot 100 - 20\cdot k\cdot(1 + k)^2 - 20\cdot k\cdot(1 + k) - 20\cdot k.$$

[7] Image source: http://www.hammacher.com/

5. Show that when $t = n$, the temperature of the beverage is:

$$T(n) = (1 + k)^{n} \cdot 100 - \frac{20k \cdot \left[1-(1+k)^{n}\right]}{1-(1+k)}.$$

HINT: Let a, r and n be numbers. The sum:

$$a + a \cdot r + a \cdot r^{2} + ... + a \cdot r^{n}$$

is equal to:

$$\frac{a \cdot \left(1 - r^{n+1}\right)}{1-r}.$$

You will have opportunities to discover why this is the case in lab on Tuesday and in section on Wednesday.