Homework Assignment 4: Solutions

1.   Your objective in this question was to create a differential equation and initial condition that describe the quantity (milligrams, mg) of the drug *Naropin*[®][1] in a patient's body. At the end of the question, you were asked to substitute the function:

$$R(T) = \frac{7200}{99} - \frac{7200}{99} \cdot e^{-0.385 \cdot T}$$

into your differential equation and initial condition to validate of your work.

Setting Up the Differential Equation

In the description of the situation, the independent variable *T* and the function *R*(*T*) were defined by the statement:

> "Let *T* represent the number of hours that a patient has been on the pain pump and *R*(*T*)
> the quantity (in milligrams, mg) of Naropin[®] in the patient's body."

Also offered was the "prototype" differential equation:

rate of change of Naropin[®] in patient's body	=	rate that Naropin[®] enters body	−	rate that Naropin[®] leaves body

In terms of the symbols defined above, the rate of change of Naropin[®] in the patient's body can be expressed as the derivative of the function *R*(*T*). Symbolically, this is: $R'(T)$.

Naropin[®] enters the patient's body when it is supplied by the pain pump. According to the description given in Homework 4:

> "A pain pump delivers a drug to the patient at a measured and very steady rate.
> According to a study reported in the *British Journal of Anaesthesiology*[2], when
> Naropin[®] is supplied at a rate of 28 mg per hour, 90% of patients reported their pain
> management to be 'good' or 'excellent.'"

Therefore, assuming that this patient is like most people, and the physician attending this case wants the patient to experience 'good' or 'excellent' pain relief, you could use the rate of 28 mg per hour for the rate at which Naropin[®] enters the patient's body.

Finally, the rate at which Naropin[®] leaves the patient's body is described by the *linear law of pharmacokinetics*. According to the description given in Homework 4:

---

[1] "Naropin" is a registered trademark of AstrZeneca International.

[2] Source: G. Turner, D. Blake and M. Buckland. (1996) "Continuois extradural infusion of ropivacaine for prevention of postoperative pain after major orthopaedic surgery." *British Journal of Anaesthesiology*, **76**: 606-610.

Therefore, the rate at which Naropin® leaves the body is equal to 0.385 times the amount of Naropin® currently in the patient's body. Symbolically this is given by: $0.385 \cdot R(T)$.

Putting all of this together with the "prototypical" differential equation gives the differential equation:

$$R'(T) = 28 - 0.385 \cdot R(T).$$

Setting up the Initial Condition

According to the description from Homework 4, the situation that you were asked to describe symbolically involved:

"...an adult patient who has just undergone major surgery. Assume that during surgery, a drug other than ropivaraine was employed as an anesthetic. Immediately after the surgery, however, the patient is placed on a pain pump supplying ropivaraine in the form of Naropin®."

In other words, when $T = 0$ the patient had no Naropin® in their body. Symbolically this statement could be expressed as:

$$R(0) = 0.$$

Validating the Differential Equation and Initial Condition with the Given Formula for $R(T)$.

The differential equation and initial condition obtained are:

$$R'(T) = 28 - 0.385 \cdot R(T) \qquad \text{and} \qquad R(0) = 0,$$

and the formula that you were given for $R(T)$ was:

$$R(T) = \frac{7200}{99} - \frac{7200}{99} \cdot e^{-0.385 \cdot T}.$$

Checking to see whether the differential equation is consistent with this formula for $R(T)$, you would substitute $R(T)$ into the left and right sides of the differential equation and determine whether you obtained the same quantity or not.

Left Side: $\qquad R'(T) \;=\; -\dfrac{7200}{99} \cdot e^{-0.385 \cdot T} \cdot (-0.385)$

---

[3] Source: http://healthanswers.telstra.com/drugdata/appco/00097100.asp   Please note that the constant of proportionality given here is only valid for individuals over the age of 12. For children, the constant of proportionality is lower than 0.385.

Right Side:

$$28 - 0.385 \cdot R(T) \quad = \quad 28 - 0.385 \cdot \left[\tfrac{7200}{99} - \tfrac{7200}{99} \cdot e^{-0.385 \cdot T}\right]$$

$$= \quad 28 - 0.385 \cdot \tfrac{7200}{99} - \tfrac{7200}{99} \cdot e^{-0.385 \cdot T} \cdot (-0.385)$$

$$= \quad -\frac{7200}{99} \cdot e^{-0.385 \cdot T} \cdot (-0.385).$$

As you obtain the same quantity by plugging the given formula for $R(T)$ into both sides of the differential equation, the given formula for $R(T)$ is compatible with the differential equation.

To show that the given formula for $R(T)$ is consistent with the initial value, plug $T = 0$ into the formula and see if you get zero. Performing this operation:

$$R(0) = \frac{7200}{99} - \frac{7200}{99} \cdot e^{-0.385*0} = \frac{7200}{99} - \frac{7200}{99} = 0.$$

Overall Comments on Problems 2-5

The starting point for Problems 2-5 is the idea that:

$T(t + 1) = T(t) +$ (change in temperature in the minute between time t and time $t + 1$).

The change in temperature (i.e. ΔT) that occurs during one minute (i.e. $\Delta t = 1$) is given by Newton's Law of Cooling. Using the numbers given for the Hammacher-Schlemmer cup, the change in temperature between time t and time $t + 1$ is:

$$\Delta T = k \cdot (T(t) - 20).$$

So, the equation for $T(t + 1)$ will be:

$$T(t + 1) = T(t) + k \cdot (T(t) - 20) = (1 + k) \cdot T(t) - 20k.$$

2. In Homework Assignment 4, you were given that $T(0) = 100$. Using the formulas developed above:
$$T(1) = (1 + k) \cdot 100 - 20k.$$

3. So far we have:
$$T(t + 1) = (1 + k) \cdot T(t) - 20k,$$
and:
$$T(1) = (1 + k) \cdot 100 - 20k.$$

Putting these two together gives:

$$T(2) = T(1) + k \cdot (T(1) - 20) = (1 + k)^2 \cdot 100 - (1 + k) \cdot 20k - 20k.$$

4.   Substituting $t = 3$ and the equation for $T(2)$ established in Question 3 into

$$T(t + 1) = (1 + k) \cdot T(t) - 20k$$

gives:

$$T(3) = T(2) + k \cdot (T(2) - 20) = (1 + k)^3 \cdot 100 - (1 + k)^2 \cdot 20k - (1 + k) \cdot 20k - 20k.$$

5.   The pattern that the calculations in Problems 2-4 have established is that:

$$T(n) = (1 + k)^n \cdot 100 - (1 + k)^{n-1} \cdot 20k - (1 + k)^{n-2} \cdot 20k - \ldots - (1 + k) \cdot 20k - 20k,$$

where n is a positive integer.

So, $T(n)$ consists of the quantity $(1 + k)^n \cdot 100$ minus a geometric series. The initial value of the geometric series (a) is:

$$a = 20 \cdot k$$

and the multiplicative factor for the geometric series (r) is:

$$r = (1 + k).$$

Using the summation formula for a finite geometric series gives that:

$$20k + 20k \cdot (1 + k) + \ldots + 20k \cdot (1 + k)^{n-2} + 20k \cdot (1 + k)^{n-1} = \frac{20k \cdot \left[1 - (1 + k)^n\right]}{1 - (1 + k)}.$$

Using this summation formula in the expression for $T(n)$ given above yields:

$$T(n) = (1 + k)^n \cdot 100 - \frac{20k \cdot \left[1 - (1 + k)^n\right]}{1 - (1 + k)}.$$