Homework Assignment 6: Solutions

1. If you had to borrow 90% of $179,000, then you would need to borrow $161,100 (=0.9*179,000). If you are not charged any interest, then all that you have to do is to repay $161,100 over the course of 30 years. In 30 years there will be 12*30 = 360 months, so the monthly repayment will be:

$$\text{Monthly Repayment} = \frac{\$161{,}100}{360} = \$447.50.$$

2. Again, if you had to borrow 90% of $179,000, then you would need to borrow $161,100. At 8% interest compounded monthly for 30 years, this would add up to:

$$161100 \cdot \left(1 + \frac{0.08}{12}\right)^{12*30} = 1761746.048.$$

Of course, the total amount of money that you have to pay to the bank over the course of the 30 year mortgage should be substantially less than $1,761,746.05 for during the mortgage you reduce the amount of the loan that is still outstanding. Remember that the idea with this question is to do a relatively easy calculation that will give us a quantity that is sure to be an over-estimate of the actual monthly payment for the mortgage.

Taking the figure of $1,761,746.05 and dividing by 360 (12 months times 30 years) gives a monthly repayment of:

$$\text{Monthly Repayment} = \frac{\$1{,}761{,}746.05}{360} = \$4893.74.$$

NOTE: The point of performing the calculations in Questions 1 and 2 was to provide a lower bound ($447.50 from Question 1) and an upper bound ($4893.74 from Question 2) for the actual mortgage repayment. The reason for doing this is: Mortgage calculations are notoriously complicated and tricky. There are a lot of places where you can inadvertently make a small error. However, the nature of the calculations involved in working out monthly repayments mean that even small errors may result in huge changes to the size of your final answer - the actual amount of money that a person would have to repay to the bank each month. Having a lower bound and an upper bound for your final answer will give you a rough guide to how big the monthly repayment should be, and can help to alert you to computational errors in your work. For example, in this homework assignment, the size of the monthly repayment is somewhere between $447.50 and $4893.74 per month. If you were to calculate a final answer of 72 cents or a final answer of $150,000 per month, then you would know that you had made a mistake somewhere.

3. In order to establish a pattern, you can write out the outstanding balances for first few months after loan was obtained.

Months after loan obtained	Outstanding balance
1	$161100\cdot\left(1+\frac{0.08}{12}\right)-M$
2	$161100\cdot\left(1+\frac{0.08}{12}\right)^2-M\cdot\left(1+\frac{0.08}{12}\right)-M$
3	$161100\cdot\left(1+\frac{0.08}{12}\right)^3-M\cdot\left(1+\frac{0.08}{12}\right)^2-M\cdot\left(1+\frac{0.08}{12}\right)-M$

The pattern that appears to be established here might be described as an initial term of the form:

$$161100\cdot\left(1+\tfrac{0.08}{12}\right)^n$$

(where n is a positive integer that is equal to the number of months since the loan was obtained) minus a geometric series of the form:

$$M+M\cdot\left(1+\tfrac{0.08}{12}\right)+M\cdot\left(1+\tfrac{0.08}{12}\right)^2+\ldots+M\cdot\left(1+\tfrac{0.08}{12}\right)^{n-1}.$$

Following this pattern, after 1 year (i.e. 12 months), the amount of the outstanding balance will be:

$$161100\cdot\left(1+\tfrac{0.08}{12}\right)^{12}-M\cdot\left(1+\tfrac{0.08}{12}\right)^{11}-M\cdot\left(1+\tfrac{0.08}{12}\right)^{10}-\ldots-M\cdot\left(1+\tfrac{0.08}{12}\right)^{2}-M\cdot\left(1+\tfrac{0.08}{12}\right)-M$$

4. With this pattern established, the only difference between the final answer to Question 3 and this answer will be that in this case the number of months, n, since the loan was taken will be equal to 360. The amount of the outstanding balance will then be:

$$161100\cdot\left(1+\tfrac{0.08}{12}\right)^{360}-M\cdot\left(1+\tfrac{0.08}{12}\right)^{359}-M\cdot\left(1+\tfrac{0.08}{12}\right)^{358}-\ldots-M\cdot\left(1+\tfrac{0.08}{12}\right)^{2}-M\cdot\left(1+\tfrac{0.08}{12}\right)-M$$

5. After 30 years, the outstanding balance should be paid down to zero. Setting the expression from Question 4 equal to zero:

$$161100\cdot\left(1+\tfrac{0.08}{12}\right)^{360}-M\cdot\left(1+\tfrac{0.08}{12}\right)^{359}-\ldots-M\cdot\left(1+\tfrac{0.08}{12}\right)^{2}-M\cdot\left(1+\tfrac{0.08}{12}\right)-M=0$$

Your ultimate goal in Question 5 is to find the value of the monthly repayment, M. to do this, you will have to solve the equation given above to make M the subject of the equation. A good first step whenever trying to solve for something like M is to re-arrange your equation so that every term involving M is on one side of the equation and every term that does not involve M is on the other side of the equation. Doing this:

$$161100 \cdot \left(1+\tfrac{0.08}{12}\right)^{360} = M + M \cdot \left(1+\tfrac{0.08}{12}\right) + M \cdot \left(1+\tfrac{0.08}{12}\right)^{2} + \ldots + M \cdot \left(1+\tfrac{0.08}{12}\right)^{359}.$$

The right hand side of this equation is a geometric series. The initial value of the geometric series is:

$$a = M$$

and the multiplicative factor of the geometric series is:

$$r = \left(1+\tfrac{0.08}{12}\right).$$

There are a total of 360 terms added together in the geometric series. Using the summation formula for a geometric series transforms the right hand side of the equation from a lot of individual terms added together to the **closed form**:

$$161100 \cdot \left(1+\tfrac{0.08}{12}\right)^{360} = \frac{M \cdot \left(1-\left(1+\tfrac{0.08}{12}\right)^{360}\right)}{1-\left(1+\tfrac{0.08}{12}\right)}.$$

The steps that remain in the calculation are now to re-arrange this equation to make *M* the subject, and then evaluating the numerical value of *M*. Re-arranging the equation to make *M* the subject gives:

$$M = \frac{161100 \cdot \left(1+\tfrac{0.08}{12}\right)^{360} \cdot \left[1-\left(1+\tfrac{0.08}{12}\right)\right]}{1-\left(1+\tfrac{0.08}{12}\right)^{360}}.$$

Evaluating this quantity with the aid of a calculator gives that the monthly repayment for the mortgage is:

$$M = \$1182.09.$$

Double-Check: Does a monthly repayment of $1182.09 seem reasonable? Although you probably don’t have a lot of experience with mortgage calculations, you can check to see whether the result falls between the lower bound ($447.50 per month) and the upper bound ($4893.74). This one does, and although that is not definite evidence to show that $1182.09 has the be the “right answer,” the fact that your careful calculation gave a number that was between the two estimates is at least reassuring.