Homework Assignment 8: Solutions

Recall that questions 1-5 of this assignment always refer to the infinite series:

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right).$$

1. The completed table is given below. These values were obtained using a TI-83 graphing calculator. The commands[1] used for each of the sums are given below.

Number of terms added	5	10	100	500	999
Total so far	0.8333	0.9090	0.9909	0.9980	0.999

- First 5 terms: `sum(seq((1/K-1/(K+1)),K,1,5))`
- First 10 terms: `sum(seq((1/K-1/(K+1)),K,1,10))`
- First 100 terms: `sum(seq((1/K-1/(K+1)),K,1,100))`
- First 500 terms: `sum(seq((1/K-1/(K+1)),K,1,500))`
- First 999 terms: `sum(seq((1/K-1/(K+1)),K,1,999))`

2. The sum that you would need to evaluate in order to work out the total of the first ten terms is:

$$\tfrac{1}{1}-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{6}-\tfrac{1}{7}+\tfrac{1}{7}-\tfrac{1}{8}+\tfrac{1}{8}-\tfrac{1}{9}+\tfrac{1}{9}-\tfrac{1}{10}+\tfrac{1}{10}-\tfrac{1}{11}=1-\tfrac{1}{11}.$$

The main simplification that is possible here is to cancel out all but two of the fractions that show up when you write out the sum. This leaves "1" and the very last fraction (i.e. $\frac{1}{11}$).

3. Based on the pattern noticed in Question 2 when simplifying, if the first N terms were added, then the "total so far" would be given by "1" minus the very last fraction. That is:

$$1-\frac{1}{N+1}.$$

[1] To see where to find these commands on a TI-83, refer to the Appendix included on the course web site with Assignment 8.

4. An important point to take from this question is that a handful of values of the "total so far" (or "partial sum") does not tell you anything definitive about the sum of an infinite series. However, calculating some numerical values for the "total so far" (or "partial sum") can give you some clues about convergence or divergence of an infinite series.

To really determine whether an infinite series converges or diverges, you often have to come up with a formula for the "total so far" (or "partial sum") and then use the methods of analysis that you learned in Math Xa to determine what the limit of the "total so far" is as the number of terms added together $N \to \infty$.

Infinite series can be quite deceptive. They can give every appearance of settling down to a finite total and still add up to an infinite total. One example of such a deceptive series is the *harmonic series* that we will investigate further when you have studied the topic of integration in more depth. The harmonic series is the infinite series:

$$\sum_{k=1}^{\infty} \frac{1}{k}.$$

If you make a table showing the "total so far" for the harmonic series, you will get something like the table shown below.

Number of terms added	5	10	100	500	999
Total of harmonic series so far	2.28333	2.928968	5.1873775	6.792823	7.4844708

Although this table does not "settle down" to a steady value quickly (like the table in Question 1 appears to), the "total so far" of the harmonic series is not growing very rapidly, either. Based on this, you might come to believe that the harmonic series adds up to a finite result. When you learn more about the calculus topic of integration, we will put that knowledge to use to show that the sum of the harmonic series is actually $+\infty$. The main point in this discussion is that a table of values of the "total so far" can often give you some clues as to whether or not an infinite series converges or diverges. However, a table of values can also deceive you. Ultimately, the only way to know for sure that an infinite series converges or diverges is to investigate the limit of the "total so far" as the number of terms added $N \to \infty$.

With that precautionary note out of the way, here is an answer to Question 4 from the homework.

For an infinite series, the meanings of the words **convergent** and **divergent** are:

• A **convergent series** is one that adds up to a finite, reasonable number. In terms of the limiting value of the "total so far," this means that as $N \to \infty$, the "total so far" gets closer and closer to a certain, finite value.

• A **divergent series** is one that does not "add up." This can happen in one of two ways:

> **1.** The "total so far" could approach an infinite value (i.e. $+\infty$ or $-\infty$) as $N \to \infty$.
>
> **2.** A second possibility is that as $N \to \infty$ the "total so far" could simply fail to approach a value at all - that is, it could bounce back and forth between two values without ever settling down to approach a definite limiting value.

If you made a table of the "total so far" (or "partial sum") as more and more terms are added together and you saw that the "total so far" was settling down to a steady value then this would be a clue to suggest that maybe the infinite series was a convergent infinite series. However, the behavior of a few numerical values of the "total so far" is not definitive evidence of convergence.

Likewise, if you made a table of "total so far" and either saw the total getting bigger and bigger, or else saw the total bouncing back and forth, these would be good reasons to suspect that the series might not converge. Again, the behavior of a few numerical values of the "total so far" is not conclusive evidence for divergence of the infinite series, either.

At the moment, your best for deciding whether a given infinite series converges or diverges is to try to find an equation for the "total so far" (or "partial sum") and then investigate the limit of the equation (using the methods of limit analysis that you learned in Math Xa) to decide what the limit is as the number of terms added $N \to \infty$. The way to use this limit is as follows:

• If the limit of the "total so far" (or "partial sum") exists and is a finite number, then the infinite series **converges**. The sum of the infinite series (also called the value that the series converges to) is equal to the value of this limit.

• If the limit of the "total so far" (or "partial sum") exists but is either $+\infty$ or $-\infty$ then the infinite series **diverges**.

• If the limit of the "total so far" (or "partial sum") does not exist then the infinite series **diverges**.

5. In the case of the series in this homework, the "total so far" when the first N terms of the series have been added is given by:

$$\textit{Total so far} = 1 - \frac{1}{N+1}$$

To investigate the limit of this as $N \to \infty$, you must consider what the expression will "look like" when extremely large values of N are present in it. When N is incredibly large:

$$\textit{Total so far} = 1 - \frac{1}{Very\_Big\_Number} = 1 - (Very\_Small\_Number) \approx 1$$

So, as $N \to \infty$ the "total so far" gets closer and closer to 1, which is a finite, sensible number. Therefore, the series in this homework assignment converges to 1. In other words, the total that you obtain by adding together an infinite number of terms, each of which has the form:

$$\frac{1}{k} - \frac{1}{k+1}$$

is:

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1.$$