

Homework Assignment 9: Solutions

1. The suggested interpretation for the expected value of the total of two dice is given below. (Remember that in Homework 5, the expected value of the total of two dice was calculated and found to be equal to 7.)

Suppose that you threw the dice many times and recorded the total that you got each time. If you were to average all of the numbers that you had recorded, then you would get an average that was very close to 7.

Translating this into the context of a naval aviator attempting to land on an aircraft carrier, you might give an interpretation like this:

Suppose that the pilot (over the course of his or her career) made many carrier landings and that the number of attempts that he or she needed to land was recorded each time. If you were to average all of the numbers that had been recorded, then you would get a result that is very close to the expected value of the number of attempts to land that the aviator needs to make.

Although you were not required to do so, you could also express this expected value with an equation resembling:

$$\text{Expected value} = \frac{\text{Total_Number_of_Attempts_in_Career}}{\text{Total_Number_of_Landings_in_Career}}.$$

Given this interpretation of the expected value, I would imagine that the expected value would be a finite number. The reasons that I have for imagining that this would be the case are:

- Even if the naval aviator has a very long career, he or she will make only a finite number of carrier landings. (This might be a pretty big number, but it will be finite nevertheless.)
- Even a minimally competent carrier-qualified aviator manages to land on his or her first few attempts in the vast majority of cases. (Remember from Homework 5 that there was a 75% chance of landing on the first attempt, a 93.75% chance of landing on the first two attempts, etc.) Therefore, every time a naval aviator attempts to land on a carrier, I would expect them to successfully land after at most several (but never a lot) of attempts.

Therefore, I would imagine that the total number of attempts to land that a naval aviator makes during his or her career would be a finite number. Provided that he or she has actually made at least one successful landing during his or her career, the quotient of the number of attempts over the number of landings will be a finite number.

In terms of the behavior of the infinite series:

$$\sum_{k=1}^{\infty} k \cdot (0.75) \cdot (0.25)^{k-1}$$

I would anticipate that (if this infinite series does a reasonable job of representing the situation of landing on an aircraft carrier) as this infinite series is supposed to be the expected value of the number of attempts to land, then the infinite series should add up to a finite result and therefore **converge**.

2. One way to mathematically demonstrate that the infinite series:

$$\sum_{k=1}^{\infty} k \cdot (0.75) \cdot (0.25)^{k-1}$$

converges is to use the **Ratio test**. When you apply the ratio test, you can organize your work into the following four steps. This is sometimes helpful to keep you sure of what you're doing, and to make sure that you don't accidentally forget an important step in the application. The four steps are:

- Step 1:** Identify the "general term" of the series.
Step 2: Find the quotient of subsequent "general terms."
Step 3: Calculate the limit of this quotient as $k \rightarrow \infty$.
Step 4: Interpret the limit using the table given below.

Value of the Limit	Interpretation
$0 \leq \text{Limit} < 1$	The infinite series converges.
Limit > 1	The infinite series diverges.
Limit = 1	The ratio test is inconclusive, so you need to try something else.

Performing these steps with the infinite series $\sum_{k=1}^{\infty} k \cdot (0.75) \cdot (0.25)^{k-1}$:

- Step 1:** The general term of the series is: $k \cdot (0.75) \cdot (0.25)^{k-1}$

- Step 2:** The quotient of subsequent general terms is:

$$\frac{(k+1) \cdot (0.75) \cdot (0.25)^k}{k \cdot (0.75) \cdot (0.25)^{k-1}} = \frac{(k+1) \cdot (0.25)^k}{k \cdot (0.25)^{k-1}} = \frac{(k+1) \cdot (0.25)}{k} = \frac{0.25 \cdot k + 0.25}{k}$$

Step 3: The limit of this quotient as $k \rightarrow \infty$ can be determined using exactly the same kind of reasoning that you would have used in Math Xa to decide whether a rational function had a horizontal asymptote or not. That is, you can compare the highest powers of k in the numerator and denominator. If the powers are equal then the limit as $k \rightarrow \infty$ is just the ratio of these highest powers. In this case, the highest power of k in the numerator is k^1 and the highest power of k in the denominator is also k^1 . Therefore the limit of the quotient is:

$$\begin{aligned} \text{Limit of Quotient} &= \frac{0.25 \cdot k}{k} \\ &= 0.25 \end{aligned}$$

Step 4: As the limit of the quotient is greater than or equal to zero and less than 1, the interpretation is that the infinite series:

$$\sum_{k=1}^{\infty} k \cdot (0.75) \cdot (0.25)^{k-1}$$

converges.

3. In this problem, the infinite series that you were asked to investigate was:

$$\sum_{k=1}^{\infty} \frac{1}{k!}$$

where the symbol $k!$ represents the product of all of the positive integers from 1 up to k . That is:

$$k! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (k-1) \cdot k$$

When trying to analyze the convergence/divergence of an infinite series like this, your best first line of defense is probably the n^{th} term test for divergence. Put simply, the n^{th} term test for divergence asserts that if the “general term” of the series approaches a non-zero limit as $k \rightarrow \infty$ then the infinite series must diverge.

In this case the general term of the series is: $\frac{1}{k!}$. As $k \rightarrow \infty$, the denominator of this

fraction becomes enormous, so $\frac{1}{k!} \rightarrow 0$ as $k \rightarrow \infty$. Remember that this **does not**

mean that the infinite series converges. After all, the n^{th} term test for divergence is a test for divergence not convergence - the n^{th} term test does not tell you anything at all about the convergence of an infinite series.

Perhaps the next best line of defense is to try to apply the **ratio test** to the infinite series. The steps in doing this are shown below.

Step 1: The general term of the infinite series is: $\frac{1}{k!}$.

Step 2: The quotient of two subsequent general terms is given below.

$$\frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{k!}{(k+1)!} = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot (k-1) \cdot k}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (k-1) \cdot k \cdot (k+1)} = \frac{1}{(k+1)}$$

Step 3: The limit of the quotient (as $k \rightarrow \infty$) is equal to zero.

Step 4: The interpretation of this limit (zero) is that the infinite series converges.

Therefore, the final answer to this problem is that the infinite series $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges.

4. In this problem, the infinite series that you were asked to investigate was:

$$\sum_{k=1}^{\infty} \frac{(1.01)^k}{1+k^2}.$$

There are two ways that you could analyze this infinite series to try to decide whether it converges or diverges. If you are good at calculating limits as $k \rightarrow \infty$ then you could establish that the limit of the “general term” is:

$$\lim_{k \rightarrow \infty} \frac{(1.01)^k}{1+k^2} = +\infty.$$

With this result, the n^{th} term test for divergence establishes that the infinite series

$$\sum_{k=1}^{\infty} \frac{(1.01)^k}{1+k^2} \text{ must diverge.}$$

A second way that you could have tackled this problem would be to use the ratio test. Very briefly, the steps involved in performing the ratio test for this infinite series are as follows.

Step 1: The “general term” of the infinite series is: $\frac{(1.01)^k}{1+k^2}$.

Step 2: The ratio of subsequent general terms is:

$$\frac{\frac{(1.01)^{k+1}}{1+(k+1)^2}}{\frac{(1.01)^k}{1+k^2}} = \frac{(1.01)^{k+1}}{(1+(k+1)^2)} \cdot \frac{(1+k^2)}{(1.01)^k} = (1.01) \cdot \frac{(1+k^2)}{(1+(k+1)^2)} = (1.01) \frac{(1+k^2)}{(2+2 \cdot k+k^2)} = \frac{1.01+1.01 \cdot k^2}{2+2 \cdot k+k^2}$$

Step 3: The limit of this ratio as $k \rightarrow \infty$ will be the ratio of the powers of k^2 in the numerator and the denominator. Therefore:

$$\begin{aligned} \text{Limit of quotient} &= \frac{1.01 \cdot k^2}{k^2} \\ &= 1.01 \end{aligned}$$

Step 4: As the limit of the quotient is greater than 1, the infinite series must diverge.

5. In this problem, the infinite series that you were asked to investigate was:

$$\sum_{k=1}^{\infty} \frac{k}{2 \cdot k + 1}.$$

Testing the “general term” of the series with the n^{th} term test as the “first line of defense” gives that:

$$\lim_{k \rightarrow \infty} \frac{k}{2 \cdot k + 1} = \frac{1}{2}.$$

As the “general term” of the infinite series approaches a non-zero limit as $k \rightarrow \infty$, the infinite series $\sum_{k=1}^{\infty} \frac{k}{2 \cdot k + 1}$ must diverge.

One final note regarding Problem 5. You might have been tempted to skip the n^{th} term test and apply the ratio test to this series immediately. If you had done so, you would have obtained that the limit of the quotient is equal to 1 as $k \rightarrow \infty$. In such a case the ratio test is inconclusive and tells you nothing about the convergence or divergence of the infinite series. Despite the fact that the ratio test is useless for deciding whether this particular series converges or diverges, the n^{th} term test easily shows that the series diverges.