Practice Problems: Exam 1

Important Information:

1. The first exam in Math Xb will be given from **7:00-9:00p.m.** on Thursday March 7. The exam will be given in **Science Center Lecture Hall D**.

2. The test will include seven to nine problems (each with multiple parts).

3. You will have 2 hours to complete the test.

4. You may use your calculator and one page (8" by 11.5") of notes on the test. You are allowed to write on both sides of the page of notes (and on the edges as well if you want).

5. The specific topics that will be tested are:

    • Quantities defined by their rates of change.
    • Slope fields and equilibrium solutions.
    • The idea of a symbolic solution for a differential equation.
    • Euler's method for approximating functions defined by their rate.
    • Finite geometric series.
    • Applications of finite geometric series: Mortgage calculations.
    • Infinite geometric series and the idea of convergence.
    • Infinite series in general.
    • The idea of a convergence/divergence test. n^{th} term and ratio tests.
    • Series, limits and area under a curve.

6. We have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the exam will resemble these problems in any way whatsoever.

7. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.

8. Good sources of help:

    • Section leaders' office hours (posted on Xb web site – click on the "Instructors" button).
    • Math Question Center (open Sunday to Thursday, 8:00-10:00 p.m. in Loker Commons).
    • Course-wide review on Wednesday 3/6 from 7:30-9:30 p.m. both in Science Center Lecture Hall E. Lecture Hall E is in the basement of the Science Center.
    • Optional labs on Tuesday 3/5 and Thursday 3/7. The schedule for labs and the people who will be leading them is shown below in case you would like to attend more than one lab.

Time	Location	Leader
10:00 a.m.	SC 111	Erin Aylward
11:00 a.m.	SC 111	Glen Yang
noon	SC 111	Chris Harrington
1 p.m.	SC 111	Anne Hamel
7 p.m.	SC 304	Bret Barnett

Figure 1: The mushroom cloud fro a nuclear test conducted in Nevada during the 1950's.

1.   Perhaps the most destructive weapon yet developed is the atomic bomb (see Figures 1[1] and 2 [2]). These weapons derive their immense destructive power from two unexpected sources. The first is an equation published by then Swiss Patent Clerk Albert Einstein, $E = mc^2$. This equation essentially says that energy (E) and mass (m) are equivalent. The energy released by an atomic explosion comes from converting minute amounts of matter into pure energy. The second unexpected source is geometric series. In this problem you will use geometric series to calculate the amount of energy released by the explosion of a nuclear weapon and calculate how many tons of TNT would be required to create an explosion of similar destructive force.

There are two basic types of nuclear weapons, **fission** and **fusion** devices. Fission weapons generate a destructive amount of energy through the splitting of large atomic nuclei into smaller nuclei. This process releases the energy that powers the nuclear explosion. **Fusion** weapons create conditions under which small nuclei (such as hydrogen) are forced together to form heavier nuclei, which is accompanied by the release of energy.

Figure 3[3] illustrates the steps in the fission of the radioactive material uranium-235 (U-235). An important condition for the correct functioning of a fission weapon is the creation of a "chain reaction." That is, the destruction of each uranium nucleus must bring about the destruction of yet further nuclei. This is achieved in a U-235 weapon by the three neutrons that are released when the uranium nucleus decays into (usually) barium and krypton.

In this problem, you will calculate the "yield" (that is, the destructive energy released) by the very first nuclear weapon, which was detonated in White Sands, New Mexico, on July 16, 1945 (see Figure 4[4]). This weapon (nick-named the "gadget") was based on a 100 kg mass of uranium (U-235, mass number = 235). Each nuclear disintegration[5] releases approximately 2.884×10^{-11} joules of energy. On average, each nuclear disintegration of U-235 leads to 2.52 subsequent nuclear disintegrations[6]. The average amount of time between a nuclear disintegration and the subsequent disintegrations that are caused is about 0.00000001 seconds (this is called the "generation time").

Figure 2: The Japanese city of Hiroshima after the first atomic bomb was dropped on August 6. 1945.

(a)   A nuclear explosion begins with just one nuclear disintegration. Use the information given above to complete the table shown below regarding the amount of energy released by the explosion of a uranium weapon.

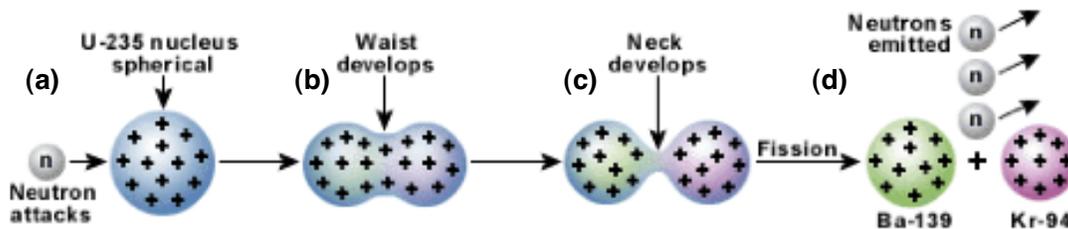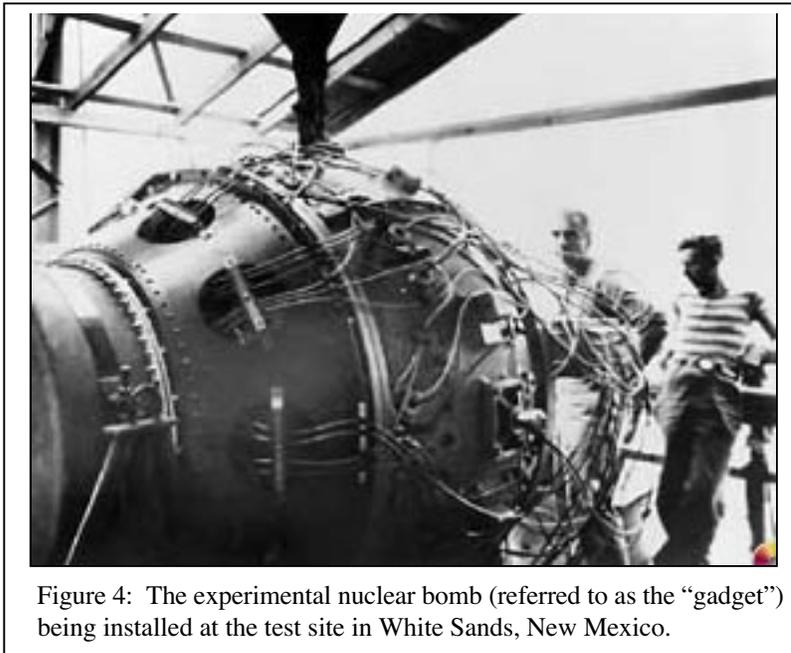| Time since nuclear explosion begun (seconds)	Total energy released (joules)
0	2.884×10^{-13}
0.00000001	
0.00000002	
0.00000003	
0.00000004	

Figure 3: Steps in the process of nuclear fission. (a) A slow moving neutron strikes a uranium nucleus. (b) The imapct of the neutron makes the nucleus unstable, and the spherical shape of the nucleus changes. (c) The shape of the uranium nucleus is further deformed until the nucleus closely resembles two separate spheres joined by a narrow "neck." (d) The "neck" is severed creating two new nuclei (typically barium and krypton). As the "neck" is severed three neutrons and some energy are released. These neutrons will go on to collide with other nuclei (causing them to undego fission).

(b)   Let N be a positive integer. Find a formula that will give the total energy released by a uranium weapon after N multiples of the generation time have passed.

Figure 4: The experimental nuclear bomb (referred to as the "gadget") being installed at the test site in White Sands, New Mexico.

(c)   As you may be able to deduce (e.g. if you have taken some classes in physical chemistry), 235 grams of uranium-235 contains approximately 6.022×10^{23} nuclei. The "gadget" detonated at White Sands contained 100 kg of uranium-235. Approximately how many uranium-235 nuclei were contained in the "gadget" ?

(d)   Assuming that all of the uranium-235 nuclei in the "gadget" were eventually struck by a neutron and made to disintegrate, about how long did the chain reaction in the "gadget" last for?

(e)   In reality, chain reactions in nuclear weapons do not last long enough to cause the disintegration of every uranium nucleus in the weapon. This is because as nuclei are broken apart, it becomes harder and

harder for the neutrons to find fresh uranium nuclei to strike and break apart. Experts[7] estimate that the "gadget" detonated in New Mexico would have created a chain reaction that lasted for about 60 generations. Determine the total amount of energy released by the "gadget."

(f)   One ton of the chemical explosive TNT will release about 4.2×10^9 joules of energy[8]. How many tons of TNT would be required to create an explosion with as much destructive energy as the "gadget" detonated in New Mexico?

(g)   If it were somehow possible to achieve a chain reaction that caused the nuclear disintegration of every uranium-235 nucleus in the "gadget," how many tons of TNT would be required to create an explosion with the same amount of destructive energy?

Figure 5: The USS Ohio (SSBN-726) returns to her home base near Bangor, WA. This photograph was taken on March 12, 1998 by Phtographers Mate 3rd Class Shawn Handley.

(h)   The United States currently has two main forms of nuclear deterrent. The first are the intercontinental ballistic missiles (ICBMs) maintained by the US Air Force. The second main nuclear deterrent are the 18 submarines of the Ohio class (see Figure 5[9]. Each Ohio class fleet ballistic missile submarine[10] carries 24 Trident-II C-4 SLBMs (sea launched ballistic missiles). Each Trident-II missile carries eight (8) multiple independently targeted re-entry vehicles (MIRVs). Each MIRV carries a 100 kiloton W76 Mark 4 nuclear warhead. This means that each W76 Mk 4 warhead can release as much destructive energy as 100,000 tons of TNT. All told, each of the 18 Ohio class submarines carries more destructive power than all of the weapons (including the atomic bombs that devastated the Japanese cities of Hiroshima and Nagasaki) that were used during World War 2. Assuming that the W76 Mark 4 is a uranium-235 weapon[11], how long (in seconds) would the chain reaction have to last for the W76 Mk 4 to deliver a 100 kiloton blast?

[7] Source: C. Sublette (1999) "Nuclear weapons FAQ." Available on-line from http://www.fas.org/nuke/hew/

[8] Source: S. Glasstone (ed.) "The Effects of Nuclear Weapons." Washington, DC: US Atomic Energy Commission, 1964.

[9] Image source: US Navy.

[10] Source: US Navy. The weapons deployment described in the problem is a bit of a simplification. 10 of the Ohio class submarines now carry the newer Trident-II D5 SLBM. The D5 SLBM carries the W-88 warhead which has a nominal yield of 480 kilotons of TNT.

[11] The warheads in a modern thermonuclear device such as the W76 Mark 4 are very complicated compared to the "gadget." The W76 Mark 4 incorporates a neutron source that is independent of the uranium-235 to increase the number of generations that the nuclear chain reaction persists for. The W76 is not purely a fission device, either. An initial fission reaction (using plutonium Pu-239, beryllium and a mixture of Deuterium and Tritium (H^2 and H^3) gas) powers a subsequent fusion reaction (using lithium deuteride as the fusion fuel) that releases vast amounts of energy from a very compact device. Each of the MIRVs in a Trident-II missile is about 6 feet long and about two feet in diameter. Source: House Report 105-851 (House Select Committee on PRC theft of US thermonuclear warhead design information.) Chapter 2, pp 68-78.

2. In the classic season 4 episode "Selma's Choice" (episode 9F11) of the TV show "The Simpsons," Marge's sister (see Figure 6[12] Selma takes Bart and Lisa to visit the amusement park "Duff Gardens." One of the attractions of Duff Gardens is the Duff Beer-A-Mid. This is a gigantic pyramid made entirely out of empty Duff beer cans. The based of the Beer-A-Mid is a square 15 meters square, and the Beer-A-Mid is 10 meters tall (see Figure 7 below).

Figure 6: Ms. Selma Bouvier

Height = 10m

15m

15m

Figure 7: A schematic picture of the Duff Beer-A-Mid.
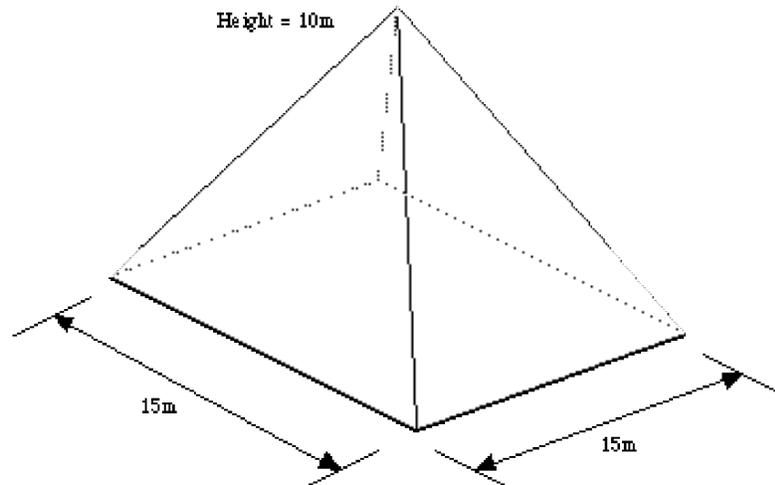
In this problem you will use a series to calculate the number of empty Duff cans that were used to build the Beer-A-Mid. In the first part of the calculation, you will make a rough estimate of the number of cans in the Beer-A-Mid so that you will have a rough idea of the numbers involved.

(a) The volume of a pyramid is equal to one third times the height times the area of the base. A standard 355 ml (12 fluid ounce) can is a cylinder of height 12.4 cm and radius 3.2 cm. Use this information to estimate the number of cans in the Duff Beer-A-Mid.

(b) The Beer-A-Mid is built up of layers of beer cans. Each layer is one can high. The height of a standard 355 ml (12 fluid ounce) can is 12.4 cm. How many layers does the Beer-A-Mid have?

(c) Use the diagram shown in Figure 4.2 (below) and the principle of similar triangles to find an equation for the width, w, of the Duff Beer-A-Mid at a height of d meters off the ground.
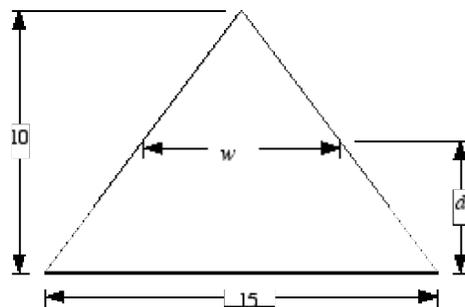

10

w

d

15

Figure 4.2: Use the principle of similar triangles to find an equation for w as a function of the height above the ground, d.

(d)   A standard 355 ml (12 fluid ounce) can has a diameter of 6.4 cm. If a layer of the Duff Beer-A-Mid is d meters off the ground, find an equation for the number of cans in that layer of the Beer-A-Mid.

(e)   Formulate a series that gives the total number of cans that were used to construct the Duff Beer-A-Mid.

(f)   Evaluate the series that you formulated in Part (d) to calculate the number of cans used to construct the Beer-A-Mid.

(g)   Was the estimate made in Part (a) an over-estimate or an under-estimate? Explain why the estimate made in Part (a) was too large or too small.

3.   The eastern gray squirrel (*Sciurus carolinensis*) is a food storing animal. During the spring, summer and fall gray squirrels collect and store food in caches. During the winter, when food is more scarce, the squirrels locate their food caches and use the food stored there to survive. In addition to the food that she needed for day-to-day survival, a particular squirrel (nicknamed "Cracky" by the biologists studying her) managed to gather and store 13 nuts each day. The biologists regularly checked Cracky's main food cache. They found that Cracky stored 13 nuts per day, but that because the food cache was in a damp tree, about 1% of the nuts went rotten each day. When she noticed a rotten nut, Cracky was observed to remove the nut from the cache and discard it.

Figure 9: Cracky the squirrel.

(a)   Assume that Cracky starts out with no nuts. According to the description given above, how many nuts will she have after 3 days?

(b)   How many nuts will Cracky have in her cache after 7 days?

(c)   Find an equation that will give the number of nuts that Cracky will have in her cache after n days.

(d)   Based on studies under laboratory conditions, the biologists estimated that Cracky would need 600 nuts to survive the winter. As soon as winter arrives, the nuts stop going bad because the weather is cooler and less humid, and the tree dries out.. How many days should Cracky spend foraging before the onset of winter?

4.   A series is defined by the formula given below,

$$\sum_{k=1}^{\infty}\left(9^{\frac{1}{k}} - 9^{\frac{1}{1+k}}\right).$$

(a)   Find an equation for the total that you get when you add the first n terms of this series.

(b)   Use the equation that you found in Part (a) and (when possible) your calculator to complete the table given below.

Number of terms added	10	100	500	999	10,000
Sum from equation					
Sum from calculator					

(c)   Tables like the one you have made in Part (b) can be used to decide whether an infinite series converges or not. Explain how this can be done.

(d)   Does the series given above converge? Justify your answer. If the series does converge, find the number that it converges to.

5[13].   Suppose that, having analyzed a series $\sum_{k=0}^{\infty} a_k$, you have been able to prove that the sum of the first n terms of the series is given by the formula:

$$S_n = \frac{16 - \frac{1}{n}}{5 + \frac{1}{n^2}}.$$

There is no need for you to try and explain or derive the formula for S_n - you can take it as a given.

(a)   Does the series $\sum_{k=0}^{\infty} a_k$ converge or not? If you believe that it does, calculate the sum of the series. If you don't believe that the series converges, explain why not. If you believe that more information is needed, list the extra information that you would need and explain how you would use that information.

(b)   Is it possible, using only the information given in this problem, to determine the value of a_k as $k \to \infty$? If you believe that it is possible to find this limit using only the information given in this problem, explain how and calculate the limiting value of a_k. Otherwise, list the extra information that you would need to know in order to calculate this limiting value.

6.   A function, $f(x)$, is defined to be:

$$f(x) = x + x^2 + x^3 + x^4 + x^5 + \ldots$$

(a)   The domain of f is all numbers between -1 and 1 (not including -1 and 1). Use the ratio test for convergence/divergence to explain why the domain of the function is this particular set of numbers.

(b)   Find an equation that will allow you to find the value of the function $f(x)$ in a much more straight-forward way than actually adding up all of the terms in the definition of $f(x)$ given above.

(c)   Use the equation that you have found in Part (b) to show that: $f'(x) = \dfrac{1}{(1-x)^2}$.

(d)   Use the ratio test for convergence/divergence to decide whether or not the series:

$$\sum_{k=1}^{\infty} \frac{k}{2^{k-1}} = 1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \ldots$$

converges or diverges. If the series converges, find the sum that this series converges to.

(e)   Use the ratio test for convergence/divergence of series to find all values of x for which the series:

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[13] This problem is based on one originally devised by Professor Lewis Blake of Duke University.

$$\sum_{k=1}^{\infty} \frac{(x-7)^k}{k^2}$$

converges.

7. For each of the infinite series given below, use the Ratio test to decide whether the series converges or diverges. If the Ratio Test proves to be inconclusive, use your calculator to make an informed guess as to whether the series converges or diverges.

(a) $\sum_{k=1}^{\infty} \frac{1}{k!}$.

(b) $\sum_{k=1}^{\infty} \frac{3^k}{k^2+1}$.

(c) $\sum_{k=1}^{\infty} \frac{1}{k}$.

(d) $\sum_{k=1}^{\infty} \frac{2^k}{k!}$.

(e) $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$.

Figure 10: A small *Codium Tomentosum* plant. The "blob" in the center of the picture is a small rock that the alga was anchored to.

8. The marine alga (i.e. seaweed) *Codium tomentosum* (see Figure 10[14]) is also known by the common names of "velvet horn," "spongy weed" and "green sea finger." Extracts from the alga are used in anti-aging skin creams[15] and as an experimental drug in AIDS research[16]. There is some concern over the fate of this plant in the UK as there is evidence to suggest that *Codium tomentosum* is being replaced by the less valuable introduced species *Codium fragile*[17].

This alga grows by dichotomous brachiation - i.e. each branch of the alga grows to a certain size and then it splits into two new branches. Each branch of the alga grows to a volume of about 24 cubic centimeters before splitting into two new branches[18].

(a) Draw some diagrams that show the appearance of a very young *Codium tomentosum* just before the first, second, third and fourth "splits" that the alga undergoes.

(b) Use the drawings that you made in Part (a) to help you fill in the first few entries of the table given below. When you have filled in the first few entries, look for a pattern in the table and generalize to write down an expression for the volume of the alga just before the Nth split.

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[14] Image source: http://www.horta.uac.pt/species/Algae/Codium_tometosum/Codium_tomentosum.htm

[15] Source: http://www.scotland.gov.uk/

[16] Source: http://www.spirulinasource.com/library-antiviral.html

[17] Source: W.F. Farnham. "Studies on aliens in the marine flora of southern England." in J.H. Price, D.E.G. Irvine and W.F. Farnham (eds.) *The Shore Environment. Volume 2: Ecosystems*. London, England: Academic Press, 1980.

[18] Source: P.E. Dixon and L.M. Irvine. *Seaweed of the British Isles*. London, England: The British Museum, 1977.

Just before *Codium tomentosum* has undergone this many splits ...	The total volume of the alga is ... (cubic centimeters)
1	24
2	
3	
4	
N	

(c)   Write down a convenient formula (or closed form) for the total volume of the alga just before the Nth split.

(d)   Chrondoritin sulfate is an experimental AIDS medication[19]. It is not clear whether this drug has any significant clinical effects in the case of AIDS patients. The active ingredients in this compound are glucuronic acid and N-Acetyl galactosomine. This compound is available in packages of 120 tablets. Each tablet contains 600 mg of glucuronic acid and the entire packet of 120 tablets costs $22.19. One cubic centimeter of *Codium tomentosum* contains about 0.001 mg of glucuronic acid[20]. How many cubic centimeters of *Codium tomentosum* would be required to supply all of the glucuronic acid on one package of Chrondoritin sulfate tablets?

(e)   Remember that each branch of *Codium tomentosum* has a volume of about 24 cubic centimeters. If all of the glucuronic acid for one package of Chrondoritin sulfate came from a single specimen of *Codium tomentosum*, how many splits would the alga have undergone?

9.   The ultimate point of this problem is to calculate the monthly payment for a mortgage. Throughout this problem, you should assume that this is an American mortgage – that means that the interest on the outstanding balance will be compounded monthly. Likewise, throughout the calculation (except for Part (e)) assume that the mortgage is a fixed-rate mortgage. That is, the interest rate will remain constant throughout the entire length of the mortgage.

(a)   Suppose that a person borrowed $200,000 at an annual interest rate of 7%. Let *M* represent the monthly mortgage payment (in dollars). Use this information to complete the table shown below.

Months since loan obtained	Amount still owed
1	
2	
3	
N	

(b)   Suppose that the person has taken out a 30 year mortgage. Calculate the monthly payment, *M*, that the individual would have to send the the lending institution.

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[19] The details given here were obtained from: http://www.aidstreatment.org/ProdList.html
[20] Source: M.E. Lai, V. Scotto and A. Bergel. "Analytical characterization of natural marine biofilms." *Proceedings of the 10th Annual Conference on Marine Corrosion and Fouling.* (Melbourne, Australia. February 7-12, 1999.)

(c)   Suppose that the person has taken out a 15 year mortgage. Calculate the monthly payment, M, that the individual would have to send the the lending institution.

(d)   If you have done the calculations in Parts (b) and (c) correctly, then you should have calculated that the monthly payment for the 15 year mortgage is considerably higher than the monthly payment for the 30 year mortgage. How much does the person end up paying the lender in each case?

(e)   Suppose that instead of a fixed-rate interest mortgage, the person has obtained a variable rate interest mortgage. This means that the interest rate can vary throughout the term of the mortgage. Generally speaking, there will be periods when interests rates are low and periods when interest rates are high. Is it more advantageous to the person borrowing the money to have low interest to begin with and high interest at the end of the mortgage or vice versa? Briefly explain your reasoning.

10.   Use convergence tests (such the nth term or ratio test) to decide which series converge and which series diverge. If both the nth term and Ratio tests are inconclusive, use a calculator to make an informed guess as to whether the series converges or diverges.

(a) $$\sum_{k=1}^{\infty} \frac{k!}{(2k)!}$$

(b) $$\sum_{k=1}^{\infty} \frac{2^{k+1}}{7^k}$$

(c) $$\sum_{k=1}^{\infty} \frac{k!}{k^2}$$

(d) $$\sum_{k=1}^{100} \frac{3+f(k)}{k^2 \cdot (k+1)}$$

where $f(j)$ is a function whose domain is all real numbers and whose range is the closed interval $[-1, 1]$.

Brief Answers. (These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.)

1.(a)   The completed table is shown below.

Time since nuclear explosion begun (seconds)	Total energy released (joules)
0	2.884×10^{-11}
0.00000001	$2.884 \times 10^{-11} + (2.884 \times 10^{-11}) \cdot (2.52)$
0.00000002	$2.884 \times 10^{-11} + (2.884 \times 10^{-11}) \cdot (2.52)$ $+ (2.884 \times 10^{-11}) \cdot (2.52)^2$
0.00000003	$2.884 \times 10^{-11} + (2.884 \times 10^{-11}) \cdot (2.52)$ $+ (2.884 \times 10^{-11}) \cdot (2.52)^2 + + (2.884 \times 10^{-11}) \cdot (2.52)^3$
0.00000004	$2.884 \times 10^{-11} + (2.884 \times 10^{-11}) \cdot (2.52)$ $+ (2.884 \times 10^{-11}) \cdot (2.52)^2 + (2.884 \times 10^{-11}) \cdot (2.52)^3$ $+ (2.884 \times 10^{-11}) \cdot (2.52)^4$

1.(b)   After N multiples of the generation time have passed, the total number of joules of energy released by the chain reaction will be given by the geometric series:

$$2.884 \times 10^{-11} + (2.884 \times 10^{-11}) \cdot (2.52) + (2.884 \times 10^{-11}) \cdot (2.52)^2 + \ldots + (2.884 \times 10^{-11}) \cdot (2.52)^N$$

Using the summation formula for a geometric series to find the total of this geometric series gives:

$$\textit{Total number of joules of energy released} \quad = \quad \frac{2.884 \times 10^{-11} \cdot \left[1 - (2.52)^{N+1}\right]}{1 - 2.52}.$$

1..(c) In 100 kg of U-235 there will be $\frac{100,000}{235} = 425.532$ moles of uranium-235. Each mole of uranium-235 contains 6.022×10^{23} nuclei (you may recognize this figure as Avogadro's number). Therefore, 100 kg of uranium-235 will contain about $425.532 \times 6.022 \times 10^{23} = 2.563 \times 10^{26}$ nuclei

1.(d) To determine how long the chain reaction in the "gadget" could have theoretically lasted for, we need to solve the equation:

$$\frac{\left[1 - (2.52)^{N+1}\right]}{1 - 2.52} = 2.563 \times 10^{26}$$

to determine the number of generations, N, that were possible in the chain reaction. The total time that the chain reaction lasted would then be the generation time (0.00000001 seconds) multiplied by the number of generations (N) in the chain reaction. Solving the equation for N:

$$1 - (2.52)^{N+1} = (2.562 \times 10^{26}) \cdot (1 - 2.52) = -3.895 \times 10^{26}$$

so that:

$$(2.52)^{N+1} = 3.895 \times 10^{26}.$$

Taking logs of both sides and re-arranging to make $N + 1$ the subject of the equation:

$$N + 1 = \frac{\log(3.895 \times 10^{26})}{\log(2.52)} = 66.244$$

Therefore, if it were possible to sustain the chain reaction so that all of the uranium-235 nuclei were disintegrated, the chain reaction would last for about 65 generations, or 0.00000065 seconds.

1.(e) To determine the total amount of energy (in joules) released by the "gadget" you can substitute $N = 60$ into the equation from Part (b). Doing this:

$$\textit{Total number of joules of energy released} \quad = \quad \frac{2.884 \times 10^{-11} \cdot \left[1 - (2.52)^{61}\right]}{1 - 2.52}$$

$$= \quad 5.802 \times 10^{13} \text{ joules.}$$

1.(f) Each ton of TNT will release about 4.2×10^9 joules of energy when detonated. The number of tons of TNT that would be required to release the same amount of destructive energy as the "gadget" would be:

$$\textit{Tons of TNT} \quad = \quad \frac{5.802 \times 10^{13}}{4.2 \times 10^9} = 13,814.51.$$

1.(g)   If the chain reaction in the "gadget" were continued for 65 generations, then we showed earlier that all of the uranium-235 nuclei would undergo disintegration. The energy released by such a chain reaction would be:

$$Total\ number\ of\ joules\ of\ energy\ released\quad = \quad \frac{2.884 \times 10^{-11} \cdot \left[1 - (2.52)^{66}\right]}{1 - 2.52}$$

$$= \quad 5.896 \times 10^{15}\ joules.$$

This would release as much destructive energy as $\frac{5.896 \times 10^{15}}{4.2 \times 10^{9}} \approx 1,403,906.13$ tons of TNT.

1.(h)   One ton of TNT will produce about 4.2×10^{9} joules of energy when detonated. Therefore, a 100 kiloton blast will produce about 4.2×10^{14} joules of energy. Setting the equation for the amount of energy released by a uranium-235 weapon equal to this figure and solving for N will give the number of generations that the chain reaction must include to produce this much destructive energy.

$$\frac{2.884 \times 10^{-11} \cdot \left[1 - (2.52)^{N+1}\right]}{1 - 2.52} = 4.2 \times 10^{14}$$

Solving this for N gives: $N = 62.14$. So, the chain reaction should last for about 62 generations (or about 0.00000062 seconds) generations to produce a 100 kiloton blast.

2(a)   A simple estimate may be found by calculating the total volume of the pyramid, calculating the volume of one can, and then dividing the volume of the pyramid by the volume of a can.

- Volume of pyramid = $(15^2)*(10)/3 = 750$ m^3.
- Volume of can = $\pi*(0.032)^2*0.124 = 0.0003989$ m^3.

So, there are roughly 750/0.0003989 = 1,880,138 cans in the Duff Beer-A-Mid.

2.(b)   The number of steps will be the height of the pyramid divided by the height of a can. This is 10/0.124 = 80.65. Rounding up, the Duff Beer-A-Mid will have 81 steps.

2.(c)   The equation relating w to d is: $w = 1.5*(10 - d)$.

2.(d)   The number of cans along the side of a step can be calculated by dividing the width of the step by the diameter of a can. This gives: $(1.5/0.064)*(10 - d) = 234.375*(10 - d)$. The number of cans in a step is the square of this quantity, i.e. $54931.64063*(10 - d)^2$.

2.(e)   The series is simply the sum of the number of cans in each of the 81 steps of the Beer-A-Mid:

$$Number\ of\ cans\ = \sum_{k=0}^{80} 54931.64063 * (10 - 0.124 * k)^2\ .$$

2.(f)   Evaluating the series given above (I used a calculator) gives 1,504,237 cans in the Beer-A-Mid.

2.(g)   The two main factors are: (1) in Part (a) we used the volume equation for a smooth-sided (rather than a stepped) pyramid. A stepped pyramid will have less volume than a smooth sided pyramid with the same base and height. (2) In between the cans there are air spaces. This is because the cans are round. In the calculation in Part (a), we did not take these air spaces into account.

3.(a) After 3 days, the number of nuts that Cracky has is given by:

$$13 + 13*(0.99) + 13*(0.99)^2 = 38.6113 \text{ nuts.}$$

(As you can only have a whole number of nuts, you would round down to 38 nuts.)

3.(b) After 7 days, the number of nuts that Cracky has is given by:

$$13 + 13*(0.99) + 13*(0.99)^2 + ... + 13*(0.99)^6 = 88.3150 \text{ nuts.}$$

(As you can only have a whole number of nuts, you would round down to 88 nuts.)

3.(c) After n days, the number of nuts that Cracky has is given by:

$$13 + 13*(0.99) + 13*(0.99)^2 + ... + 13*(0.99)^{n-1} = \frac{13(1-0.99^n)}{(1-0.99)}.$$

3.(d) To solve this problem, set the equation from Part (c) equal to 600 and solve for n.

$$\frac{13(1-0.99^n)}{(1-0.99)} = 600$$

$$(0.99^n) = 0.538462$$

$$n = \log(0.538462)/\log(0.99) = 61.59.$$

Therefore, Cracky will need to start collecting nuts about 62 days before the onset of winter.

4.(a) The sum of the first n terms of the series is: $9 - 9^{1/(n+1)}$.

4.(b) The completed table is given below.

Number of terms added	10	100	500	999	10,000
Sum from equation	7.778905	7.9780069	7.9956046	7.9978008	7.9997802
Sum from calculator	7.778905	7.9780069	7.9956046	7.9978008	N/A

4.(c) An infinite series is said to converge if the total that you get from adding the first n terms approaches a finite limiting value as n approaches infinity. Tables like the one made in Part (b) can help you to decide whether or not the total of the first n terms of the series is approaching a limiting value or not.

4.(d) In the case shown above in Part (b), the entries in the table seem to be getting closer and closer to 8 as n gets larger and larger. This leads me to suspect that the series does converge, and that it converges to a sum of 8.

5.(a) Yes the series converges. This is because the total of the first n terms of the series (this total is given by S_n) approaches a finite limiting value as n approaches infinity. The limiting value is 16/5, which is also the sum of the series.

5.(b) We know, from Part (a), that the series converges. Therefore, the nth term test says that as $k \to \infty$, a_k must approach zero. (Otherwise, if a_k did not approach zero, the series could not converge.) Therefore, the limiting value of a_k as $k \to \infty$ is equal to zero.

6.(a)   The function $f(x)$ is defined by a series: $f(x) = \sum\limits_{k=1}^{\infty} x^k$. The function $f(x)$ is defined whenever the series converges. So, the domain of the function $f(x)$ will be the set of x-values for which the series converges. The general term of the series is $a_k = x^k$. The limit of the ratio a_{k+1}/a_k is equal to x. So, by the ratio test, the series will converge for any x between -1 and 1 (not inclusive of the end-points).

6.(b) $f(x)$ is an infinite geometric series. Using the summation formula for an infinite geometric series gives:

$$f(x) = \frac{x}{1-x}.$$

6.(c)   Using the quotient rule to differentiate the equation for $f(x)$ found in Part (b) gives:

$$f'(x) = \frac{(1-x) + x}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

6.(d)   The ratio $a_{k+1}/a_k = 0.5*(k+1)/k$. The limit of this ration as k approaches infinity is 0.5. Therefore, by the ratio test the series converges. The series convergence to a sum of 4.

6.(e)   The ratio $a_{k+1}/a_k = (x-7)*(k-1)^2/k^2$. The limit of this ratio as k approaches infinity is $(x-7)$. According to the ratio test, the series will converge when $-1 < x - 7 < 1$, i.e. when $6 < x < 8$.

7.(a)   The general term of the series is: $\frac{1}{k!}$. The ratio of subsequent general terms is:

$$Ratio = \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{k!}{(k+1)!} = \frac{1}{k+1}$$

The limit of this ratio (as $k \to \infty$) is equal to zero, giving the conclusion that the series converges.

7.(b)   The general term of the series is: $\frac{3^k}{k^2+1}$. The ratio of subsequent general terms is:

$$Ratio = \frac{\frac{3^{k+1}}{(k+1)^2+1}}{\frac{3^k}{k^2+1}} = \frac{3^{k+1}}{\left[(k+1)^2+1\right]} \cdot \frac{\left[k^2+1\right]}{3^k} = 3 \cdot \frac{\left[k^2+1\right]}{\left[(k+1)^2+1\right]}$$

The limit of this ratio (as $k \to \infty$) is equal to three. The interpretation of this limit is that the infinite series diverges.

7.(c)   The general term of the series is: $\frac{1}{k}$. The ratio of subsequent general terms is:

$$Ratio = \frac{\frac{1}{(k+1)}}{\frac{1}{k}} = \frac{k}{k+1}$$

The limit of this ratio (as $k \to \infty$) is equal to one. The interpretation of this limit is that the ratio test is inconclusive in this particular case. A table of the "total so far" for different numbers of terms added together is given below.

Number of Terms added	10	50	100	500	999
Total so far	2.9289	4.4992	5.1874	6.7928	7.4845

It is not really very clear from this table whether the "total so far" is settling down or not. It is certainly not "shooting up" at a phenomenal rate. On the other hand, the "total so far" does increase when the number of terms added. If I had to guess, I would guess that the series diverges. However, this is just a guess and (if all that we have to go on are the entries in the table) then convergence is certainly not out of the question.

7.(d) The general term is: $\frac{2^k}{k!}$. The ratio of subsequent general terms is:

$$Ratio \;=\; \frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}} = \frac{2^{k+1}}{(k+1)!} \cdot \frac{k!}{2^k} = 2 \cdot \frac{k!}{(k+1)!} = 2 \cdot \frac{1}{k+1} = \frac{2}{k+1}$$

The limit of this ratio (as $k \to \infty$) is equal to zero. The interpretation of this limit is that the infinite series converges.

7.(e) The general term is: $\dfrac{k^2}{2^k}$. The ratio of subsequent general terms is:

$$Ratio \;=\; \frac{\frac{(k+1)^2}{2^{k+1}}}{\frac{k^2}{2^k}} = \frac{(k+1)^2}{2^{k+1}} \cdot \frac{2^k}{k^2} = \frac{1}{2} \cdot \frac{(k+1)^2}{k^2}$$

The limit of this ratio (as $k \to \infty$) is equal to one half. The interpretation of this limit is that the infinite series converges.

8.(a) Rough sketches of a *Codium tomentosum* alga just before each of the first four "splits" are shown below.

Just before first split	Just before second split	Just before third split		Just before fourth split
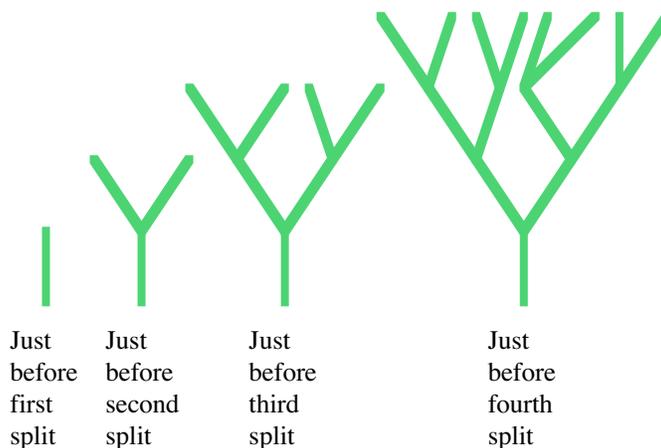
8.(b) The completed table is shown below. You may not have written out the entries of the table in quite the same way as shown below. That's fine so long as the entries you have written down are equivalent ot the entries shown below. I have written the entries in this way to try to make the pattern in the table a little more evident.

Just before *Codium tomentosum* has undergone this many splits ...	The total volume of the alga is ... (cubic centimeters)
1	24
2	$24 + 24{\cdot}2$
3	$24 + 24{\cdot}2 + 24{\cdot}2^2$
4	$24 + 24{\cdot}2 + 24{\cdot}2^2 + 24{\cdot}2^3$
N	$24 + 24{\cdot}2 + 24{\cdot}2^2 + \dots + 24{\cdot}2^{N-1}$

8.(c)   Using the summation formula for a geometric series with the last entry in the table above gives:

$$\text{Volume before } N^{th} \text{ Split} = \frac{24 \cdot \left[1 - 2^N\right]}{1 - 2} = 24 \cdot \left[2^N - 1\right].$$

8.(d)   The amount of glucuronic acid in one package of tablets is: $600{\times}120 = 72{,}000$ mg. As each cubic centimeter of *Codium tomentosum* contains 0.001 mg of glucuronic acid the total volume of *Codium tomentosum* will be: 72,000/0.001 = 72,000,000 cubic centimeters of *Codium tomentosum*.

8.(e)   To determine the number of splits that the alga had undergone, we can set the equation from Part (c) equal to 72,000,000 and solve for *N*. Performing this calculation:

$$24 \cdot \left[2^N - 1\right] = 72{,}000{,}000$$

$$2^N - 1 = \frac{72{,}000{,}000}{24} = 3{,}000{,}000$$

$$N = \frac{\log\left(3{,}000{,}000 + 1\right)}{\log(2)} = 21.517$$

So, $N = 21.517$. All of that glucuronic acid could be obtained from an alga that was just about to split for the 21st or 22nd time.

9.(a)   The completed table is shown below.

Months since loan obtained	Amount still owed
1	$200000 \cdot \left(1 + \frac{0.07}{12}\right) - M$
2	$200000 \cdot \left(1 + \frac{0.07}{12}\right)^2 - M \cdot \left(1 + \frac{0.07}{12}\right) - M$
3	$200000 \cdot \left(1 + \frac{0.07}{12}\right)^3 - M \cdot \left(1 + \frac{0.07}{12}\right)^2 - M \cdot \left(1 + \frac{0.07}{12}\right) - M$
N	$200000 \cdot \left(1 + \frac{0.07}{12}\right)^N - M \cdot \left(1 + \frac{0.07}{12}\right)^{N-1} - \dots - M \cdot \left(1 + \frac{0.07}{12}\right) - M$

9.(b)   Thirty years is equivalent to 360 months. Assuming that the individual has been responsible and has actually paid off the mortgage in those 360 months, he or she would owe nothing after 360 months.

Using the last entry from the table in Part (a) with $N = 360$, and setting the resulting expression equal to zero gives:

$$200000 \cdot \left(1 + \tfrac{0.07}{12}\right)^{360} - M \cdot \left(1 + \tfrac{0.07}{12}\right)^{359} - \ldots - M \cdot \left(1 + \tfrac{0.07}{12}\right) - M = 0$$

The idea is to now solve this expression to find the value of M. Moving all of the terms that involve M to the right hand side of the equation gives:

$$200000 \cdot \left(1 + \tfrac{0.07}{12}\right)^{360} = M + M \cdot \left(1 + \tfrac{0.07}{12}\right) + \ldots + M \cdot \left(1 + \tfrac{0.07}{12}\right)^{359}$$

Using the summation formula for a geometric series on the right hand side of this equation gives:

$$200000 \cdot \left(1 + \tfrac{0.07}{12}\right)^{360} = \frac{M \cdot \left[1 - \left(1 + \tfrac{0.07}{12}\right)^{360}\right]}{1 - \left(1 + \tfrac{0.07}{12}\right)}.$$

Re-arranging to make M the subject of the equation and evaluating all of the numerical quantities gives:

$$M = \$1330.60 \text{ per month.}$$

9.(c) If the mortgage is a 15 year mortgage instead of a 30 year mortgage then it must be paid off after 180 months. Therefore,

$$200000 \cdot \left(1 + \tfrac{0.07}{12}\right)^{180} - M \cdot \left(1 + \tfrac{0.07}{12}\right)^{179} - \ldots - M \cdot \left(1 + \tfrac{0.07}{12}\right) - M = 0.$$

Again, the idea is to solve this equation to find the value of M. Moving all of the terms that involve M to the right side of the equation gives:

$$200000 \cdot \left(1 + \tfrac{0.07}{12}\right)^{180} = M + M \cdot \left(1 + \tfrac{0.07}{12}\right) + \ldots + M \cdot \left(1 + \tfrac{0.07}{12}\right)^{179}$$

Using the geometric summation formula with the geometric series on the right hand side of the equation gives:

$$200000 \cdot \left(1 + \tfrac{0.07}{12}\right)^{180} = \frac{M \cdot \left[1 - \left(1 + \tfrac{0.07}{12}\right)^{180}\right]}{1 - \left(1 + \tfrac{0.07}{12}\right)}.$$

Re-arranging this equation to make M the subject and evaluating the numerical quantities gives:

$$M = \$1797.66 \text{ per month.}$$

9.(d) The total amounts paid to the lender in each case are shown the table below.

Number of Months	Monthly payment ($)	Total paid to lender ($)
180	1797.66	323578.50
360	1330.60	479016.00

The table shows that although the monthly payment is lower for the 30 year mortgage, the total amount that is paid to the lender is considerably higher than the total amount paid to the lender with the 15 year mortgage.

9.(e)   I would prefer to have a low interest rate at the beginning of the mortgage and a high interest rate at the end of the mortgage. This is because at the beginning of the mortgage almost all of your monthly mortgage payment goes to offsetting the interest that the outstanding balance incurs. If the interest rate is low at this point of the mortgage then more of your monthly payment will go towards reducing the outstanding balance rather than just offsetting the interest. At the end of the mortgage the situation is reversed and most of the money that you send in each month is going towards reducing the size of the outstanding balance with very little offsetting interest. Therefore, if you have to have a high interest rate during some point of your mortgage, the high interest rate would exact the least amount of money from you if it came along towards the end of the mortgage.

10.(a)   In this series the general term is: $\frac{k!}{(2k)!}$. The ratio of subsequent general terms is:

$$Ratio \; = \; \frac{\frac{(k+1)!}{(2k+2)!}}{\frac{k!}{(2k)!}} = \frac{(k+1)!}{(2k+2)!} \cdot \frac{(2k)!}{k!} = \frac{(k+1)}{(2k+1)\cdot(2k+2)} = \frac{1}{2\cdot(2k+1)}.$$

The limit of this ratio (as $k \rightarrow \infty$) is equal to zero. The interpretation of this limit is that the infinite series converges.

10.(b)   The general term of the series is: $\dfrac{2^{k+1}}{7^k}$. The ratio of subsequent general terms is:

$$Ratio \; = \; \frac{\frac{2^{k+2}}{7^{k+1}}}{\frac{2^{k+1}}{7^k}} = \frac{2^{k+2}}{7^{k+1}} \cdot \frac{7^k}{2^{k+1}} = \frac{2}{7}.$$

The limit of this ratio (as $k \rightarrow \infty$) is equal to 2/7. The interpretation of this limit is that the infinite series converges.

10.(c)   The general term of the series is: $\frac{k!}{k^2}$. The ratio of subsequent general terms is:

$$Ratio \; = \; \frac{\frac{(k+1)!}{(k+1)^2}}{\frac{k!}{k^2}} = \frac{(k+1)!}{(k+1)^2} \cdot \frac{k^2}{k!} = \frac{(k+1)\cdot k^2}{(k+1)^2} = \frac{k^2}{(k+1)}.$$

The limit of this ratio (as $k \rightarrow \infty$) is $+\infty$. The interpretation of this limit is that the infinite series diverges.

10.(d)   This is about the only trick question on this entire set of practice problems. If you look very carefully at the upper limit on the sigma notation, you will see that the upper limit of summation is 100, not ∞. This means that (d) is actually a finite series as it only has 100 terms added together. The sum of 100 finite terms will be a finite number. Hence this series adds up to a finite result, which means that the series converges.