Practice Problems: Exam 1

Important Information:

1. The first exam in Math Xb will be given from **7:00-9:00p.m.** on Thursday March 7. The exam will be given in **Science Center Lecture Hall D**.

2. The test will include seven to nine problems (each with multiple parts).

3. You will have 2 hours to complete the test.

4. You may use your calculator and one page (8” by 11.5”) of notes on the test. You are allowed to write on both sides of the page of notes (and on the edges as well if you want).

5. The specific topics that will be tested are:

   - Quantities defined by their rates of change.
   - Slope fields and equilibrium solutions.
   - The idea of a symbolic solution for a differential equation.
   - Euler’s method for approximating functions defined by their rate.
   - Finite geometric series.
   - Applications of finite geometric series: Mortgage calculations.
   - Infinite geometric series and the idea of convergence.
   - Infinite series in general.
   - The idea of a convergence/divergence test. n^{th} term and ratio tests.
   - Series, limits and area under a curve.

6. We have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the exam will resemble these problems in any way whatsoever.

7. Remember: On exams, you will have to supply evidence for your conclusions, and explain why your answers are appropriate.

8. Good sources of help:

   - Section leaders’ office hours (posted on Xb web site – click on the “Instructors” button).
   - Math Question Center (open Sunday to Thursday, 8:00-10:00 p.m. in Loker Commons).
   - Course-wide review on Wednesday 3/6 from 7:30-9:30 p.m. both in Science Center Lecture Hall E. Lecture Hall E is in the basement of the Science Center.
   - Optional labs on Tuesday 3/5 and Thursday 3/7. The schedule for labs and the people who will be leading them is shown below in case you would like to attend more than one lab.

Time	Location	Leader
10:00 a.m.	SC 111	Erin Aylward
11:00 a.m.	SC 111	Glen Yang
noon	SC 111	Chris Harrington
1 p.m.	SC 111	Anne Hamel
7 p.m.	SC 304	Bret Barnett

Figure 1: A male patient suffering from the classical pattern of hair loss due to androgenetic alopecia.

1. The scientific name for male-pattern baldness (see Figure 1[1]) is *Androgenetic alopecia*. Male pattern baldness is caused by secretion of the enzyme 5-alpha-reductase, which is produced in the prostrate gland. When the enzyme interacts with the hormone *androgen*, dyhydrotestosterone (DHT) is produced. It is the presence of significant levels of DHT in the body that is thought to cause the actual hair loss.

According to the FDA[2], Americans spend over one billion dollars per year on hair loss treatments. Among the many hair-loss treatments currently on offer[3], only two drugs (minoxidil topical solution, sold under the trade name of Rogaine® (see Figure 2[4] and finasteride, sold under the trade name of Propecia®) and one form of surgery (transplants) as effective treatments for androgenic alopecia. Believe it or not, the FDA have conducted clinical trials of more than 250,000 hair-loss treatments and these are the only three that have been able to consistently demonstrate any significant hair growth[5].

Androgenetic alopecia is estimated to affect more than 35 million American men and as many as 20 million American women. In this problem, you will study the effects of using the popular (but expensive) hair-loss treatment minoxidil. In this problem, T will represent the number of days since a person suffering from androgenetic alopecia started treating their condition with minoxidil, and $H(T)$ will represent the amount of hair that the person has on his head.

Figure 2: Packaging for the minoxidil topical solution Rogaine(R).

(a) An adult who has not experienced a significant amount of hair loss normally has about 100,000 to 150,000 hairs on his or her head[6] (gender makes no difference). People normally shed between 50 and 100 hairs per day. In a person who is not experiencing hair loss, these hairs are replaced by new hair growth. In an individual who is experiencing androgenetic alopecia, the hair does not grow back. Use the information provided here to estimate how quickly someone with androgenetic alopecia could go bald, and the longest amount of time it would take someone to go bald.

(b) Assume that the person who has androgenetic alopecia

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[1] Image source: http://www.dermatology.org/

[2] Source: Devera Pine. "Hair! From personal statement to personal problem." *FDA Consumer,* December 1991.

[3] Episode 7F02 ("Simpsons and Delilah") of the hit TV show "The Simpsons" dealt with the topic of male pattern baldness. The hair treatments showcased on that episode included: "Hair Master," "Gorilla Man Scalp Blaster," "Hair Chow," "Bald Buster," "NU GRO," "U Wanna B Hair-E" and "Hair in a Drum." Commenting on the last product, a doctor on the show advised Homer, "Of course any hair growth that you experience while using this product will be purely coincidental."

[4] Image source: http://www.tticreates.com/

[5] Some of the ineffective treatments have been based on outrageous ingredients, including dog urine, cow saliva, spider webs and a salve used by the German army in World War I for the treatment of open sores on artillery horses.

[6] Source: http://www.sn2000.com/sl/faqhair.htm

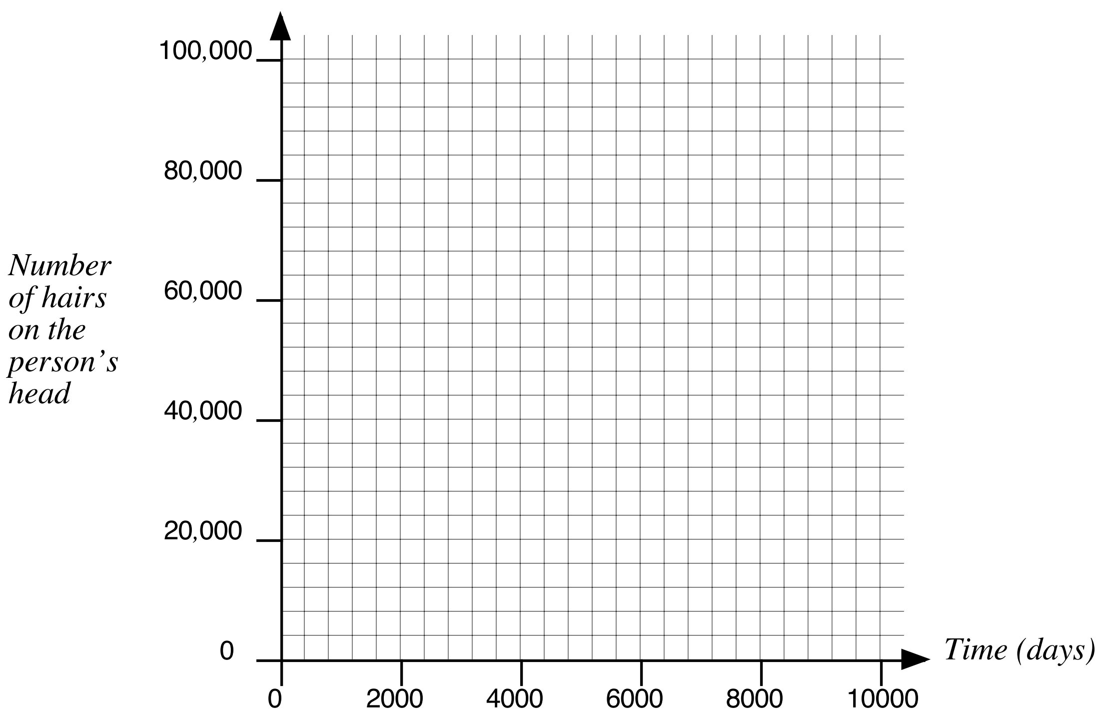
becomes aware of their condition when they still have 100,000 hairs left on their head. At this point, they immediately begin using a minoxidil topical solution (such as Rogaine®). Rogaine® normally takes about 4 months (120 days) to show any perceptible re-growth of hair. If the person is losing hair at an average rate of 75 hairs per day, how much hair will he have when the Rogaine® takes effect?

(c) Clinical trials of a topical minoxidil solution[7] suggest that after four months of application, the solution can stimulate the re-growth of hair. One interpretation of the clinical research suggests that this re-growth occurs (on average) at a rate equal to 0.09% of the number of hairs present. That is, the rate at which hair re-grows is proportional to the amount of hair present, and the constant of proportionality is about 0.0009. Set up a differential equation that gives the rate at which the amount of hairs on the person's head changes. The following prototypical equation may be a helpful starting point.

(Net Rate of change) = (Rate at which hair re-grows) – (Rate at which hair falls out)

(d) Determine any equilibrium solutions that the differential equation from Part (c) may have.

(e) Use the grid provided below to sketch the slope field corresponding to the differential equation that you found in Part (c). Label the equilibrium solution(s) that you found in Part (d) and sketch a graph showing how the amount of hair that the person has changes as time goes by. (Make sure that your graph goes through the point that you calculated in Part (b).)



(f) Describe the effects of using minoxidil topical solution that this model predicts. Do you think that these predictions are reasonable? Explain why or why not.

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[7] Source: J.L. Roberts. (1987) "Androgenetic alopecia: Treatment results with topical minoxidil." *Journal of the American Academy of Dermatology*, **16**(3): 705-710.

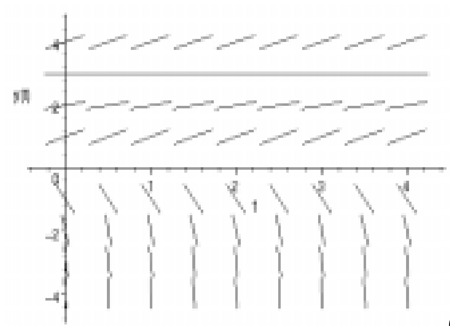
2. In this problem, the differential equation that you will be working with is alwys the differential equation:

$$\frac{dy}{dt} = \frac{1}{4} \cdot y(t) \cdot \left[y(t) + 2 \right] \cdot \left[y(t) - 3 \right]^2 .$$

(a) Find the location of any equilibrium solution(s) that this differential equation might have.

(b) Use the equation for the derivative $\frac{dy}{dt}$ to decide which of the following slope fields could possibly be the slope field that corresponds to the differential equation in this problem.

(I)



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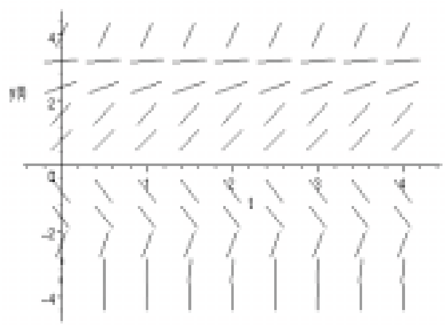

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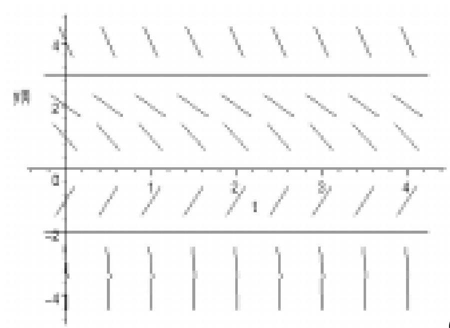

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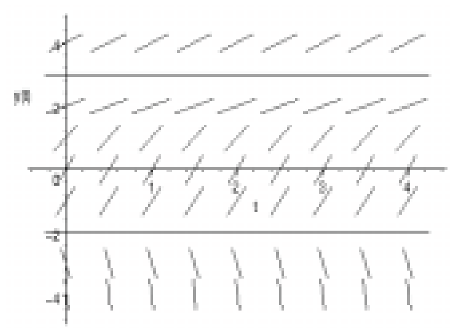

(c) Use the slope field from Part (b) to classify each of the equilibrium solution(s) that you found in Part (a) as stable, unstable or semi-stable.

(d) A slope field is shown below. Find a differential equation whose slope field would resemble the one shown below.

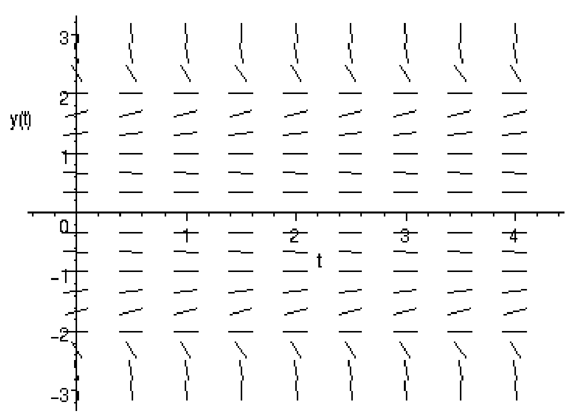

Figure 3: Dr. Judah Folkman.

3. Until recently, one of the most controversial figures in cancer research was Boston physician and researcher Dr. Judah Folkman (see Figure 3[8]). In the 1970's while working as a surgeon Dr. Folkman noticed that cancerous, malignant tumors were always connected to a plentiful blood supply that provided them with the nutrients that they needed to grow and become dangerous to the health of the patient. In a 1976 article[9]. Dr. Folkman suggested that tumors must have some sort of mechanism for encouraging nearby blood vessels to grow small off-shoots to supply the tumor with blood (see Figure 4[10]) a process known as angiogenesis. Based on his theories, Dr. Folkman attempted to develop drugs that would inhibit angiogenesis – that is, would inhibit tumors from gaining access to a blood vessel and the nutrients transported by the blood.

In his lab, Dr. Folkman and his collaborators have created the drug Endostatin (see Figure 5[11]). In animal tests (see Figure 6[12]) tumors in mice have been dramatically reduced in size through the application of endostatin. The drug is currently undergoing the first phase of trials with human patients that are required by the FDA.

In this problem, let T represent the number of hours since a patient began receiving endostatin and let $E(T)$ represent the amount of endostatin in the patient's body (measured in mg).

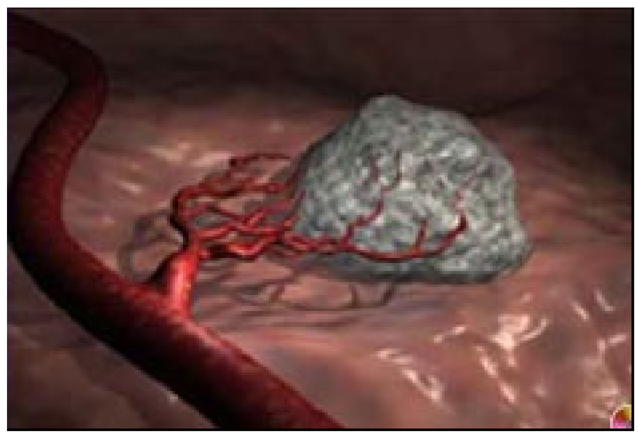

Figure 4: Angiogenesis. The tumor has been able to encourage a nearby blood vessel to gow an off-shoot. As a result the tumor now has a blood supply to provide it with the nutrients that it needs to grow (and threaten the health of the patient).

(a) Some initial trials of endostatin with human patients have not produced very encouraging results. For example, the phase I clinical trials conducted in the Dana-Farber Cancer Institute of Massachusetts General Hospital 12 of the 19 patients participating in the trial had to be taken off endostatin treatment due to progression of their cancers[13]. In general, the early phase I trials administered endostation to patients over very short intervals (like injecting the drug). More recent trials have changed the delivery method to a continuous infusion method, where the drug is supplied to the patient continuously at a very steady rate. In trials reported by the MD Anderson Cancer Center at the University of Texas[14], researchers administered endostatin to patients at a rate of 25 mg per hour. Endostatin appears to obey the *linear law of pharmacokinetics*, meaning that the rate at which it is removed from the body is proportional to the amount of endostatin in the body. There are no very good figures for the constant of proportionality, but very

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[8] Image source: http://www.fumento.com/

[9] J. Folkman. (1976) "The vascularization of tumors." *Scientific American*, **234**(59): 64-73.

[10] Image source: http://www.pbs.org/

[11] Image source: http://www.entremed.com/

[12] Image source: http://abcnews.go.com/

[13] Source: http://www.endostatin.net/endostatin/results.shtml

[14] Source: K. Muller and K.L. Wright. (2000) "Endostatin: Phase I trial yields promising preliminary results." *OncoLog*, **45**(11), November 2000.

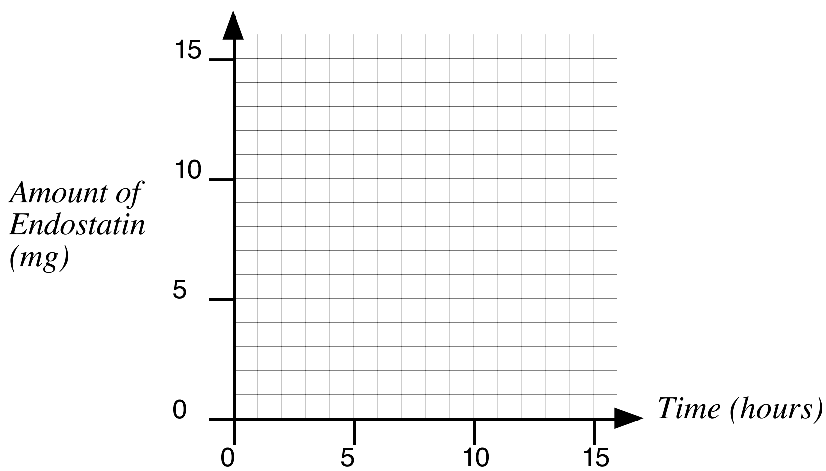
preliminary results from the Duke University[15] medical center suggest a ball-park figure of about 2.7. Use the prototypical equation given below to create a differential equation that describes the rate of change of the amount of endostatin in a patient's body.

(Rate of change of amount of endostatin) = (Rate at which endostatin enters the body) –
(Rate at wich endostatis is eliminated from the body)

Figure 5: Vials of the angiogenesis-inhibiting protein Endostatin.

(b) When beginning treatment with endostatin, patients do not have any endostatin in their body at all. Express this statement using mathematical symbols.

(c) Use the grid provided below to sketch the slope field of the differential equation that you created in Part (a). Use the slope field to sketch a curve showing how the amount of endostatin in the patient's body will change as time goes by.



(d) Locate any equilibrium solutions of the differential equation. Are the equilibrium solution(s) stable, unstable or semi-stable? Briefly explain how you can tell. Interpret the significance of any equilibrium solution(s) in terms of the amount of endostatin in the patient's body.

(e) A function $E(T)$ is given below. Determine whether or not this function is a solution of the differential equation that you created in Part (a) of this problem, and whether or not the function solves the initial condition that you formulated in Part (b) of this problem.

$$E(T) = 9.259 - 9.259 \cdot e^{-2.7 \cdot T}.$$

[15] Personal communication, 3/3/2002.

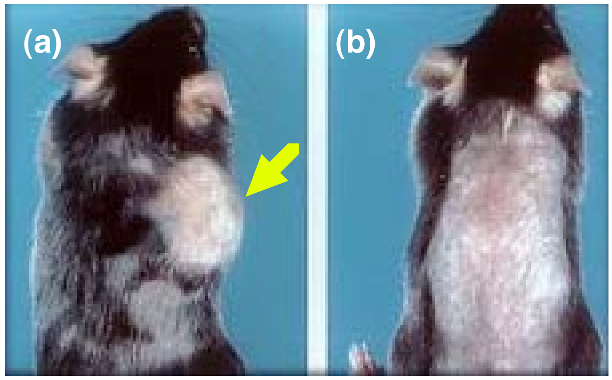


Figure 6: (a) The mouse has a large tumor. (b) After treatment with Endostatin the size of the tumor is dramatically reduced.

4. Cardiovascular diseases (diseases of the heart and blood vessels) are America's number one killer[16]. Almost 25% of the population – more than 60 million people - suffer from some form of cardiovascular disease. During 1996, deaths due to heart disease represented 41.4% of all deaths, with one death occurring (on average) every 33 seconds for the entire year[17].

Death rates due to heart disease vary according to age, gender, race and geographical location. Generally speaking, men have a higher death rate from heart disease than do women. Minorities often suffer from a higher death rate due to heart disease than does the white population. Geographically, the state with the lowest death rate due to heart disease is Utah, followed by Colorado, Hawaii, New Mexico and Alaska. Mississippi has the highest death rate due to heart disease, followed by South Carolina, Tennessee, the District of Columbia and Louisiana.

Perhaps the only good news regarding heart disease is that the death rate (per 100,000 people) has dropped steadily for the last 50 years. In 1960, for example, the national death rate was 369 per 100,000 people[18]. Studies conducted in 1995 suggested that the death rate had dropped to approximately 281 deaths (per 100,000 people). Two important factors that are thought to have contributed to the reduction of the cardiovascular death rate are the reduction in the number of people smoking, and a general reduction in the fat content of the average person's diet.

In this problem, T will represent the number of years since 1960, and $C(T)$ will represent the death rate due to heart disease at time T. The units of $N(T)$ are number of deaths for every 100,000 people living in America per year.

The function $C(T)$ obeys the differential equation:

$$C'(T) = -0.02989 \cdot \left[C(T) - 250 \right]$$

quite closely.

(a) In the introduction to this problem, it was noted that in 1960, the death rate due to heart disease was 369. Express this information using the mathematical symbols defined above.

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[16] Source: U.S. Department of Health and Human Resources.

[17] Source: U.S. Department of Health and Human Resources. *Health in the United States*. Washington, DC: Department of Health and Human Resources, 1998.

[18] Source: U.S. Department of Health and Human Resources.

(b) Use Euler's method and the table provided below to approximate the death rate due to heart disease in the year 2010.

Year	T	C(T)	$C'(T)$	$C'(T)\cdot\Delta T$	New C(T)
1960	0				
1970	10				
1980	20				
1990	30				
2000	40				

(c) Use the differential equation:

$$C'(T) = -0.02989\cdot\left[C(T) - 250\right]$$

to find an expression for the second derivative, $C''(T)$.

(d) Is the estimate that you have calculated for the death rate an over- or an under-estimate of the "true" death rate in 2010? Briefly explain how you know.

(e) Show that the function defined by the formula:

$$C(T) = 250 + 119\cdot e^{-0.02989\cdot T}$$

satisfies both the differential equation and the initial condition.

(f) Now that you have a formula that gives the "true" values of $C(T)$, how could you use this formula to decide whether the estimate (from Euler's method) of the death rate in 2010 is an over- or an under-estimate?

Figure 7: An industrial robot used to perform wleding operations in a vehicle assembly plant.

5. Industrial robots (see Figure 7[19] are a fixture in the manufacturing industry. Despite their high initial cost, robots often represent a viable option to companies because of the direct and indirect labor costs that they save. Of course, the flip-side of this argument is that the adoption of industrial robots has put a lot of people out of work.

An industrial robot like the one shown in Figure 7 has a price[20] of about $72,200, although if the robot is equipped with any specialized equipment the price can be a lot higher.

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[19] Image source: http://www.ifr.org/

[20] Source: http://reharob.manuf.bme.bu/publications/d3_1.4.pdf

One of the accounting exercises that firms engage in is to estimate the lifetime of the "log-lived assets" that a company owns, such as industrial robots. As soon as they are installed, long-lived assets tend to lose value (this is called *depreciation*). Accountants try to make models to forecast how the value of long-lived assets (such as robots) will change as time goes by. The (estimated) amount that an asset could be sold for at the end of its useful life is called the *salvage value*[21] of the asset. (For example, an asset like an industrial robot could be sold for spare parts at the end of its useful life. The amount that these spare parts could be sold for will be the salvage value of the robot.) When the value of a long-lived asset reaches the salvage value of the asset, then (at least according to accountants) the asset has reached the end of its useful life. In many accounting exercises, the salvage value is taken to be zero. However here we will assume that the salvage value will have to cover the cost of removing the robot from the work place. (This usually costs about $5000 for a robot like the one shown in Figure 7[22].

In this problem, let T represent the number of years since an industrial robot was installed, and $V(T)$ represent the value of the robot in dollars.

(a) In the introduction to this problem, you were told that when an industrial robot is first installed, the value of the robot is about $72,000. Express this information in symbolic form.

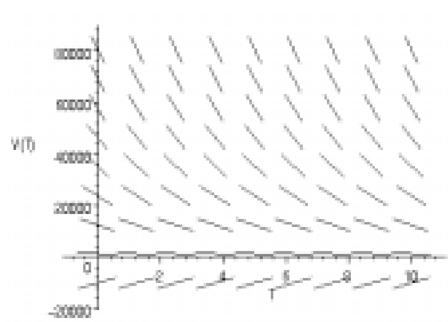
(b) Three accounting methods of calculating the depreciation on a long-term asset are described below. Express each accounting method symbolically as a differential equation.

Straight line depreciation: In this model, the value of the robot decreases by the same amount each year. In the case of the robot shown in Figure 7, the value decreases by $9025 dollars every year.

Declining balance depreciation: In this model, the rate of change of the robot's value is proportional to the value of the robot. For the robot shown in Figure 7, the constant of proportionality is −0.33375.

Declining difference depreciation: In this model, the rate of change of the robot's value is proportional to the difference between the robot's current value and the robot's salvage value ($5000). For the robot pictured in Figure 7, the constant of proportionality is −0.33375.

(c) Four slope fields are shown below. Three of these slope fields correspond to the different accounting methods that could be used to calculate depreciation and one does not correspond to any of the accounting methods. Match the slope fields to the different accounting methods.



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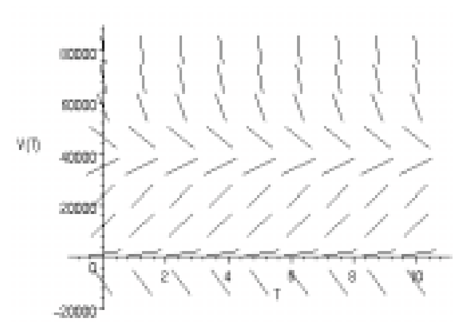

(II)

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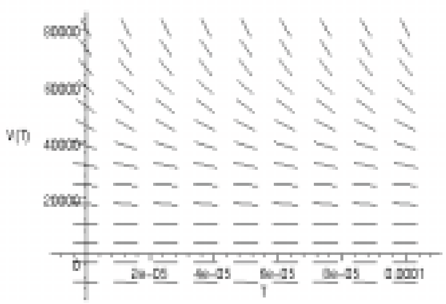
[21] Source: http://moneycentral.msn.com/

[22] Source: http://www.robotics.uc.edu/IR2001/Deters.pdf/



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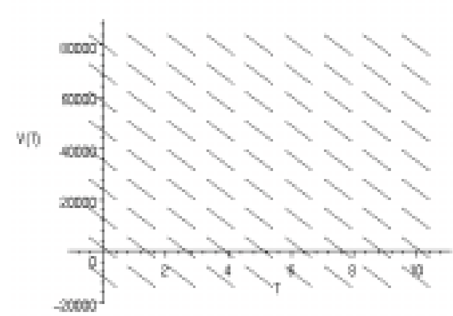


(IV)

(d) The formulas for three functions are given below. Determine which function is a solution of which differential equation.

$$V(T) = 5000 + 67200 \cdot e^{-0.33375 \cdot T}$$

$$V(T) = 72200 - 9025 \cdot T$$

$$V(T) = 72200 \cdot e^{-0.33375 \cdot T}$$

(e) For each of the three accounting methods that might be employed, determine the useful life of the industrial robot.

Brief Answers. (These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.)

1.(a) The person who would take the least amount of time to suffer complete hair loss would be the person who started with 100,000 hairs and began to lose them at a rate of 100 per day. In this case, the person would completely lose their hair in 1000 days (slightly under 3 years). The person who would take the most time to suffer complete hair loss would be the individual who started with 150,000 hairs and began to lose them at a rate of 50 per day. Such a person would have 3000 days (slightly more than 8 years) before they suffered complete hair loss.

1.(b) The person will have about 100,000 – 120*75 = 91,000 hairs on their head when the minoxidil solution begins to take effect.

1.(c) Recall that *T* represents the amount of time since the person became aware of their condition and began treatment with minoxidil solution. Likewise, recall that *H*(*T*) represents the number of hairs that the person has on his head. Then the rate of change of the amount of hair can be expressed by the symbols: $H'(T)$. The rate at which hairs re-grow is equal to 0.009 multiplied by the number of hairs, that is by *H*(*T*). Finally, the rate at which the person loses their hair is given in Part (b) as 75 hairs per day. Substituting these symbols into the prototypical differential equation gives:

$$H'(T) = 0.0009 \cdot H(T) - 75\,.$$

1.(d) The equilibrium solution(s) of a differential equation can be located by setting the derivative $H'(T)$ equal to zero, and then solving for the function *H*(*T*). Carrying this out with the differential equation from Part (c):

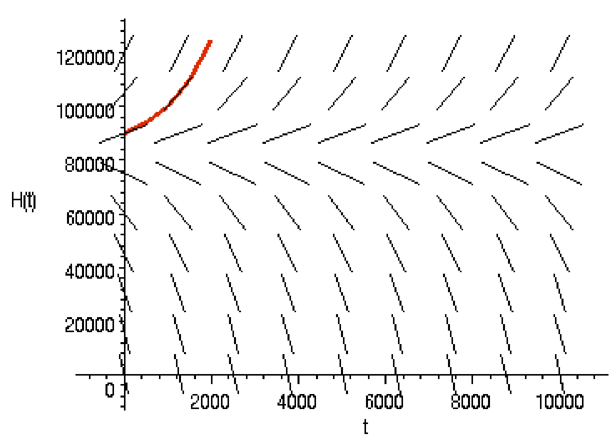
$$H'(T) = 0.0009 \cdot H(T) - 75 = 0$$

so that:

$$H(T) = \frac{75}{0.0009} = 83{,}333.33 \text{ hairs.}$$

This is the only equilibrium solution that the differential equation has.

1.(e) The slope field for the differential equation (along with solution curve) is shown below. Note that the solution curve shown goes through the point (120, 91000), which was the point predicted by the calculation in Part (b).



1.(f) If the graph shown in Part (e) is to be believed, then it appears that application of a minoxidil solution leads to phenomenal hair restoration. Moreover, it appears that once the minoxidil kicks in, hair growth really takes off. This seems quite unrealistic, as users of minoxidil would quickly come to resemble "Cousin It" from the Addams Family. Therefore, the interpretation of the clinical research that this model is based on is probably a flawed interpretation.

2.(a) The equilibrium solutions may be located by setting the derivative $\frac{dy}{dt}$ equal to zero and then finding all of the values that $y(t)$ could possibly assume to make the equation true. Carrying this out:

$$y(t)\cdot\left[y(t)+2\right]\cdot\left[y(t)-3\right]^2=0.$$

This equation will be satisfied when:

- $y(t)=0$
- $y(t)=-2$, or,
- $y(t)=3$.

2.(b) To decide which slope field corresponds to the differential equation:

$$\frac{dy}{dt}=\frac{1}{4}\cdot y(t)\cdot\left[y(t)+2\right]\cdot\left[y(t)-3\right]^2$$

you can inspect the slope fields to see whether they have equilibrium solutions in all of the places that you expect (based on your answer to Part (a)). In this case, you would expect to see equilibrium solutions at $y(t) = 0$, $y(t) = -2$ and $y(t) = 3$. Based on this, you can immediately reject the slope fields **(I)** and **(IV)** as these slope fields show only two equilibrium solutions. To decide between the slope fields **(II)** and **(III)** you can try substituting a point into the differential equation and determining the slope of the slope field at that point. With luck, only one of the two remaining slope fields will be compatible with this information. A quick look at the two slope fields reveals that when $y(t) = 2$, the slope field **(II)** has a positive slope whereas the slope field **(III)** has a negative slope. If you actually plug $y(t) = 2$ into the differential equation you get:

$$\frac{dy}{dt}=\frac{1}{4}\cdot(2)\cdot\left[2+2\right]\cdot\left[2-3\right]^2=2.$$

As the slope is positive, slope field **(II)** must be the one that corresponds to the differential equation.

2.(c) Remember that equilibrium solutions can be classified in three ways. The features of an equilibrium solution that lead to the various classifications are described in the table below.

Type of equilibrium solution	Appearance of little line segments near the equilibrium solution (when read from **left to right**)
Stable	Little line segments are attracted towards the equilibrium solution
Unstable	Little line segments are repelled away from the equilibrium solution
Semi-stable	On one side (above or below) the little line segments are attracted towards the equilibrium solution. On the other side, (below or above) the little line segments are repelled away from the equilibrium solution.

Using this classification system, the equilibrium solutions of the differential equation:

$$\frac{dy}{dt} = \frac{1}{4} \cdot (2) \cdot [2+2] \cdot [2-3]^2 = 2$$

are as follows:

- $y(t) = 0$ This equilibrium is an unstable equilibrium solution.
- $y(t) = -2$ This equilibrium is a stable equilibrium solution.
- $y(t) = 3$ This equilibrium is a semi-stable equilibrium solution.

2.(d) The differential equation that you come up with has to have the following features:

- An unstable equilibrium at $y(t) = -2$.
- A stable equilibrium at $y(t) = -1$.
- An semi-stable equilibrium at $y(t) = 0$.
- An unstable equilibrium at $y(t) = 1$.
- A stable equilibrium at $y(t) = 2$.

Working purely from the locations of the equilibrium solutions (and the fact that $y(t) = 0$ is semi-stable, suggesting that this factor will be squared in the differential equation), the differential equation should look something like:

$$\frac{dy}{dt} = k \cdot [y(t)+2] \cdot [y(t)+1] \cdot [y(t)]^2 \cdot [y(t)-1] \cdot [y(t)-2],$$

where k is a constant. To decide whether k is positive or negative, we need to have a look at the slope field and note whether the slope is positive or negative at a particular point. For example, when $y(t) = 3$ the little line segments are pointing down indicating that the derivative is negative. Substituting $y(t) = 3$ into the differential equation given above:

$$\frac{dy}{dt} = k \cdot [3+2] \cdot [3+1] \cdot [3]^2 \cdot [3-1] \cdot [3-2] = 360 \cdot k.$$

In order to have $\frac{dy}{dt} < 0$, you need $k < 0$. Therefore, any differential equation that you write down that looks like:

$$\frac{dy}{dt} = k \cdot [y(t)+2] \cdot [y(t)+1] \cdot [y(t)]^2 \cdot [y(t)-1] \cdot [y(t)-2],$$

with $k < 0$ would be a differential equation that could plausibly have the slope field shown in Question 2(d).

3.(a) The prototypical equation suggests that we need to figure out how to represent each of the following quantities using mathematical symbols such as T (the number of hours since endostatin treatment began) and $E(T)$, the amount (in mg) of endostatin in the patient's body:

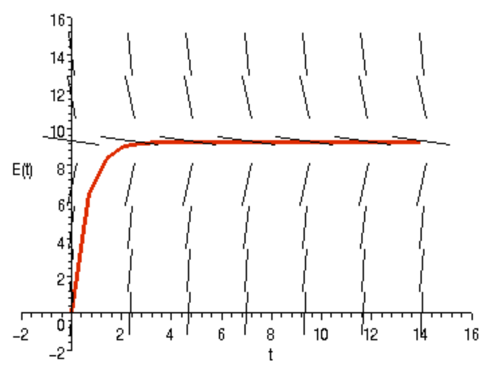
- The rate of change of the amount of endostatin the patient's body.
- The rate at which endostatin enters the patient's body.
- The rate at which endostatin is eliminated from the patient's body.

The first of these rates is the derivative, $E'(T)$, of the function $E(T)$. The value of the next rate can be inferred from the description of the Phase I clinical trials carried out at the MD Anderson Cancer Center at the University of Texas. In those trials, patients were supplied with endostatin at the steady rate of 25 mg per hour. Therefore, the rate at which the patient receives endostatin is 25 mg per hour. Finally, to establish the last rate, the preliminary pharmacokinetic studies that have been carried out with endostatin suggest that the rate at which the drug is eliminated from the body is proportional to the amount of the drug that is present. Therefore, the rate at which endostatin is eliminated from the patient's body is equal to some constant of proportionality multiplied by the function $E(T)$. Although the value of this constant of proportionality has not be definitively established, preliminary work suggests that it might be around 2.7. Hence, the rate at which endostatin is eliminated from the body is: $2.7 \cdot E(T)$. Substituting these rates into the prototypical equation gives the differential equation:

$$E'(T) = 25 - 2.7 \cdot E(T).$$

3.(b) At the beginning of the treatment the patients have no endostatin in their bodies. This situation can be described symbolically by: $E(0) = 0$.

3.(c) The slope field and solution curve are shown in the diagram below.



3.(d) To locate the equilibrium solution, set $E'(T) = 0$ and solve the resulting equation to make $E(T)$ the subject of the equation. Doing this gives that the differential equation:

$$E'(T) = 25 - 2.7 \cdot E(T)$$

has only one equilibrium solution, and that it is:

$$E(T) = \frac{25}{2.77} = 9.259 \text{ mg.}$$

From the appearance of the slope field drawn in Part (d), when read from **left to right** the little line segments appear to be attracted towards the equilibrium solution. Therefore the equilibrium is a stable equilibrium. The significance of this equilibrium is that this is the maximum amount of endostatin that will accumulate in the patient's body.

3.(e) To demonstrate that the function:

$$E(T) = 9.259 - 9.259 \cdot e^{-2.7 \cdot T}$$

is a solution of the differential equation:

$$E'(T) = 25 - 2.7 \cdot E(T),$$

you need to substitute the function into the left and right hand side of the differential equation separately and then verify that the two expressions that you get are, in fact, the same. Doing this:

Left Hand Side $= E'(T)$

$= -9.257 \cdot e^{-2.7 \cdot T} \cdot (-2.7)$

Right hand side $= 25 - 2.7 \cdot E(T)$

$= 25 - 2.7 \cdot \left[9.259 - 9.259 \cdot e^{-2.7 \cdot T}\right]$

$= -9.257 \cdot e^{-2.7 \cdot T} \cdot (-2.7)$

As the left hand side equals the right hand side, the function $E(T) = 9.259 - 9.259 \cdot e^{-2.7 \cdot T}$ is a solution of the differential equation.

To verify that the function $E(T) = 9.259 - 9.259 \cdot e^{-2.7 \cdot T}$ satisfies the initial condition (i.e. $E(0) = 0$) you need only to substitute $T = 0$ into the function and verify that you get zero as the result. Doing this:

$$E(0) = 9.259 - 9.259 \cdot e^{-2.7 \cdot 0} = 9.259 - 9.259 = 0.$$

4.(a) The information that you have to represent here is the fact that in 1960, the death rate due to heart disease was 369 people per year (for every 100,000 people living in the United States). That is, when $t = 0$, the function is equal to 369. In symbols this means: $C(0) = 369$.

4.(b) The completed table is shown below.

Year	T	C(T)	$C'(T)$	$C'(T) \cdot \Delta T$	New C(T)
1960	0	369	−3.55691	−35.5691	333.4309
1970	10	333.4309	−2.4937496	−24.937496	308.493404
1980	20	308.493404	−1.7483678	−17.48367845	291.0097255
1990	30	291.0097255	−1.225780697	−12.25780697	278.7519185
2000	40	278.7519185	−0.859394845	−8.59394845	270.1579701

Therefore, the estimate for the death rate due to heart disease in 2010 is about 270.16 deaths per year (for every 100,000 people living in the United States).

4.(c) To find an expression for the second derivative $C''(T)$, you can take the differential equation:

$$C'(T) = -0.02989 \cdot \left[C(T) - 250\right].$$

Doing this gives:

$$C''(T) = -0.02989 \cdot C'(T) = (-0.02989)^2 \cdot \left[C(T) - 250\right].$$

4.(d) To determine whether the estimate of 325.5 is an under- or an over-estimate, you can look at the concavity of the function $C(T)$. Generally speaking:

When the function is …	The estimate provided be Euler's Method is …
Concave up	An under-estimate of the "true" value of the function
Concave down	An over-estimate of the "true" value of the function

When $T = 0$, we know the "true" value of the function. It is $C(0) = 369$. Substituting this into the equation for the second derivative given above:

$$C''(0) = (-0.02989)^2 \cdot [369 - 250] = 0.1063.$$

Recall (e.g. from Math Xa) the relationship between the sign of the second derivative and the concavity of the original function (summarized in the table given below). The fact that the second derivative of the function $C(T)$ is positive gives that the function $C(T)$ is concave up. Therefore, the estimate of 270.16 given by Euler's method is likely an under-estimate of the "true" value of the function $C(T)$ for the year 2010.

When the second derivative is …	The original function is …
+	Concave up
−	Concave down

4.(e) To show that the given function:

$$C(T) = 250 + 119 \cdot e^{-0.02989 \cdot T}$$

is a solution of the differential equation, we must substitute the function into the left and right hand sides of the differential equation separately and then show that the expressions that you obtain are actually the same.

Left hand side $= C'(T)$

$= 119 \cdot e^{-0.02928 \cdot T} \cdot (-0.02928)$

Right hand side $= -0.02928 \cdot [C(T) - 250]$

$= -0.02928 \cdot [250 + 119 \cdot e^{-0.02928 \cdot T} - 250]$

$= (-0.02928) \cdot 119 \cdot e^{-0.02928 \cdot T}$

As the expressions obtained by substituting the functions into left and right hand sides of the equations separately turn out to be exactly the same, the function is a solution of the differential equation.

To show that the function satisfies the initial condition, $C(0) = 369$, you need to substitute $T = 0$ into the equation for $C(T)$ and verify that you get 369 as the result. Doing this:

$$C(0) = 250 + 119 \cdot e^{0} = 369.$$

4.(f) Now that you have a formula that gives the "true" values of $C(T)$ you could just substitute $T = 50$ (to get the death rate for the year 2010) into the formula for $C(T)$ and compare the result you obtain to the value predicted by Euler's method (which is 270.16). Evaluating $C(T)$ for $T = 50$ gives:

$$C(50) = 250 + 119 \cdot e^{-0.02928 \cdot 50} = 277.525793.$$

The estimate provided by Euler's method is certainly lower than this "true" value, so the estimate provided by Euler's method really is an under-estimate of the "true" death rate in the year 2010.

5.(a) The information be be described is that when $T = 0$, the value of the function is 72,200. This can be expressed by the symbolic statement: $V(0) = 72{,}200$.

5.(b) **Straight line depreciation:** $V'(T) = -9025$

Declining balance depreciation: $V'(T) = -0.33375 \cdot V(T)$

Declining difference depreciation: $V'(T) = -0.33375 \cdot [V(T) - 5000]$

5.(c) The slope for **straight line depreciation** does not depend on either T or $V(T)$. Therefore the slope field for straight line depreciation should always have the same slope no matter what. The only slope field with this appearance is **(IV)**.

The slope field for **decreasing balance depreciation** has an equilibrium solution at $V(T) = 0$ and no other equilibrium solutions. The only slope field that shows this behavior is **(I)**.

The slope field for **decreasing difference depreciation** has an equilibrium solution at $V(T) = 5000$ and no other equilibrium solutions. The only slope field that shows this behavior is **(III)**.

5.(d) The correspondences may be found by taking the derivative of each of the functions that are given and determining which of the three differential equations that derivative matches. The answers are:

Straight line depreciation: $V(T) = 72200 - 9025 \cdot T$

Declining balance depreciation: $V(T) = 72200 \cdot e^{-0.33375 \cdot T}$

Declining difference depreciation: $V(T) = 5000 + 67200 \cdot e^{-0.33375 \cdot T}$

5.(e) To calculate the useful life of the industrial robot in each case, you set the appropriate function $V(T)$ equal to 5000 (the salvage value of the robot) and then solve for T.

Straight line depreciation:

$$V(T) = 72200 - 9025 \cdot T = 5000$$

$$T = \frac{72200 - 5000}{9025} = \frac{67200}{9025} = 7.446.$$

So, with the straight line depreciation model, the useful life of the robot is about 7.446 years.

Declining balance depreciation:

$$V(T) = 72200 \cdot e^{-0.33375 \cdot T} = 5000$$

$$e^{-0.33375 \cdot T} = \frac{5000}{72200}$$

$$-0.33375 \cdot T = \ln\left(\frac{5000}{72200}\right)$$

$$T = \frac{\ln\left(\frac{5000}{72200}\right)}{-0.33375} = 8\,.$$

So, with the declining balance depreciation model, the useful life of the robot is about 8 years.

Declining difference depreciation:

This is about the only "trick question" on this set of practice problems. It is impossible to solve the equation:

$$5000 + 67200 \cdot e^{-0.33375 \cdot T} = 5000$$

for a finite value of T. Therefore, according to this model, the robot stays useful indefinitely.