Math Xb - Differential Equations and Series

1. **(a)** Find the solution to the initial value problem given below.

$$\frac{dy}{dx} = 3y - 4, \ y(0) = 3.$$

(b) Find the solution to the initial value problem given below.

$$\frac{dy}{dx} = 3x - 4, \ y(0) = 3.$$

(c) Explain why the two similar-looking initial value problems give such different solutions.

2. Solve the initial value problem:

$$\frac{dy}{dt} = -3y + 5 \qquad , \qquad y(1) = \frac{-1}{3}.$$

(To have a chance of full credit for a problem like this on an exam, you would be expected to show the method of solution and not simply plug values into a formula.)

3. A highly desirable (but unfortunately rare) kind of investment is a *continuously compounding* account. This is one in which the rate at which the investment changes is directly proportional to the current value of the investment. The constant of proportionality is usually called the *continuous growth rate*.

(a) Write a differential equation that describes a continuously compounding account.

(b) Suppose you invest $10,000 at a continuous rate of 13% per annum. Solve the differential equation from part (a) to find an explicit formula for the value of the investment after t year.

(c) Sketch a graph showing the value of the investment as a function of time.

(d) How long will it take for your investment to double in value?

4. A challenging hobby that has become popular in the last decade is keeping coral, shrimp, anemones and other saltwater creatures in a miniature reef aquarium. One of the most important elements in the reef aquarium is calcium, which is used by the sea creatures in their metabolism, and when growing their shells. The aquarist must add calcium to the aquarium, usually by dripping a solution of calcium hydroxide into the tank.

(a) In a small reef tank, calcium is added at the rate of 2 mg per hour. A test for the amount of calcium in the tank conducted at the start of the day indicates that there is 2g of calcium in the tank (i.e. 2000 mg). When the levels of calcium aren't critically low or critically high, the sea creatures use calcium at a rate proportional to the amount of calcium in the tank. Write a differential equation and an initial condition that represent the situation described here.

(b) When is the derivative in your differential equation equal to zero?

(c) Solve the differential equation from part (a).

(d) Describe what happens to the amount of calcium in the tank as time goes by. What level does it stabilize at?

5. A calculus professor notices that his hair is falling out. Being a calculus professor, he sets out to study the situation using the tools of calculus. Based on a close inspection of his scalp, the calculus professor reckons that about 3000 new hairs grow each year. On the other hand, the hairs seem to fall out at a rate proportional to the number of hairs on the professor's head. At present, the calculus professor has about 5000 hairs on his head.

(a) Write down a differential equation and initial condition that describe this hair-raising scenario. (Your answer may contain one unspecified constant.)

(b) Solve the differential equation that you have written down in part (a). (Your answer may contain one unspecified constant.)

(c) After a year, the professor reckons that he only has about 4800 hairs left on his head. Find the numerical value of the unspecified constant from part (b). (**Note: you may need to use a graphing calculator to answer this.)**

(d) What does the future hold for this calculus professor's head of hair?

6. A series is defined by the formula given below,

$$\sum_{k=0}^{\infty} \frac{1+2^k}{5^k}.$$

(a) Explain why this series converges.

(b) Find the number that the series converges to. Make sure that you justify any steps in your calculation.

(c) You are given a series, $\sum_{k=0}^{100} a_k$ and all that you know about it is that for every value of *k*, $0.5 < a_k < 1$. What can you say about the convergence of the series?

7. **(a)** Find the exact value of the infinite sum,

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \cdots$$

by recognizing that the sum is a geometric series.

(b) Find the infinite sum $\sum_{k=0}^{\infty} \left(\frac{1}{9}\right)^k$ and explain how you could use this result to calculate the exact value of the infinite sum in (a)

8. Wicket the Ewok is foraging for nuts to last through the severe Endor winter. Each day, Wicket manages to gather 13 nuts, but notices that each day about 1% of his stash goes bad and has to be thrown out.

(a) Assume that Wicket starts out with no nuts. How many nuts will he have after 3 days?

(b) How many nuts will Wicket have in his stash after 7 days?

(c) Find a formula that gives the number of nuts that Wicket will have in his stash after *n* days?

(d) Wicket needs 600 nuts to make it through the winter. As soon as winter arrives, the nuts stop going bad because the weather is cooler. How many days before the onset of winter should Wicket start foraging?

9. When jet fighters land on aircraft carriers, they try to snag an arrester cable with their tail hook. The cable stops them (preventing them from rolling off the side of the ship and plunging into the ocean). It is not uncommon for a fighter to miss the arrester cable and have to go around for another try - experts estimate that the airplane catches the cable about 85% of the time.

(a) Find the probability that a pilot will catch the wire on the n^{th} time around.

(b) Find the probability that a pilot will catch the wire with n or fewer attempts.

(c) Jet fighters usually carry enough fuel for about five attempts. What percentage of pilots manage to land their fighters?

10. A function, $f(x)$, is defined to be:

$$f(x) = x + x^3 + x^5 + x^7 + x^9 + \ldots$$

(a) The domain of f is all numbers between -1 and 1 (not including -1 and 1). Explain why the domain of the function is this particular set of numbers.

(b) Show that: $$f'(x) = \frac{1+x^2}{\left(1-x^2\right)^2}.$$

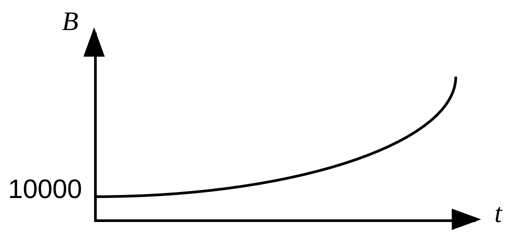
(c) Sketch an accurate graph of $y = f(x)$ for $-1 < x < 1$. Be sure to label any interesting features such as x-intercepts, maximums, minimums, points of inflection and y-intercepts.

Answers

1.(a) $y = (5/3)e^{3x} + 4/3$.
1.(b) $y = 1.5x^2 - 4x + 3$.
1.(c) The thing that makes the two equations different is the fact that the first equation has the dependent variable (i.e. *y*) on the right hand side, whereas the second equation has the independent (i.e. *x*) variable on the right hand side. To solve the differential equation in (b) you just have to ask yourself, "Hmm, what would I have to differentiate to wind up with $3x - 4$?" because the right hand side only involves the independent variable. To solve the differential equation in part (a) you have to make the substitution $w = y - 4/3$ to change the equation into $w' = 3w$, which is one you know how to solve.

2. $y = -40.17e^{-3t} + \frac{5}{3}$.

3.(a) Let $B(t)$ denote the size of the investment at time *t*. Let *r* denote the continuous growth rate. $dB/dt = r*B$.
3.(b) $B(t) = 10{,}000*e^{0.13t}$.
3.(c) See graph below.



3.(d) About 5.33 years.

4.(a) *t* = hours since calcium test performed. $C(t)$ = mg of calcium in tank. $C(0) = 2000$.
$dC/dt = 2 - k*C$, where *k* is a positive constant.
4.(b) When the amount of calcium in the tank is equal to $2/k$.
4.(c) Solution is: $C(t) = 2/k + A_0e^{-kt}$ where A_0 is a constant.
4.(d) As $t \to \infty$, this goes to $2/k$ as the exponential approaches zero.

5.(a) Let *t* be the time in years starting right now and *H* the number of hairs on the professor's head. The $dH/dt = -kH + 3000$, where *k* is a positive constant.
5.(b) $H = (5000 - 3000/k)e^{-kt} + 3000/k$.
5.(c) $k = 0.654508$.
5.(d) Eventually, the professor's head will stabilize at about 4584 hairs.

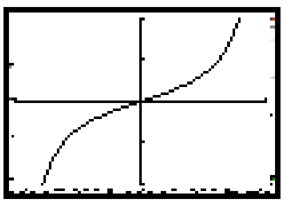
6.(a) You can break this up into two convergent geometric series added together.
6.(b) 35/12.
6.(c) Converges (because it is a finite series - it has 101 terms), and the sum is between 50.5 (=0.5*101) and 101 (=1*101).

7.(a) 3/4.
7.(b) The sum is 9/8. The sum in (a) is the sum in (b) minus one third of the sum in (b).

8.(a) 38.6113 nuts - round down to 38.
8.(b) 88.3150 nuts - round down to 88.
8.(c) $\frac{13(1-0.99^n)}{(1-0.99)}$
8.(d) About 61.59 days.

9.(a) $0.85*(0.15)^{n-1}$.
9.(b) $(0.85 - 0.85*(0.15)^n) / (1 - 0.15)$
9.(c) 99.99%

10.(a) The formula defining $f(x)$ is a geometric series. The multiplicative factor (i.e. r) is x^2. The domain of $f(x)$ will be where this series converges, which is where the multiplicative factor is less than one, i.e. $x^2 < 1$, or $-1 < x < 1$.
10.(b) Using the summation formula for an infinite geometric series, $f(x) = x/(1 - x^2)$. If you differentiate this using the quotient rule, you will get the result.
10.(c) The graph will look something like:



There are vertical asymptotes at $x = 1$ and $x = -1$. There is a point of inflection at $x = 0$. The x- and y-intercept are both at (0, 0).