Appendix C: Factoring Algebraic Expressions

"Factoring" algebraic equations is the reverse of "expanding" algebraic expressions discussed in Appendix B. Factoring algebraic equations can be a great help when trying to find solutions of equations.

Example C.1

Find all of the real numbers x that satisfy the algebraic equation:

$x^2 - 2x + 1 = 0$.

Solution:

One possible approach is just to guess a value of x and plug this guess into the equation to check. For example, if you guess $x = 2$, plugging in gives:

$2^2 - 2 \cdot 2 + 1 = 4 - 4 + 1 = 1$.

This is not equal to zero, so $x = 2$ is not a solution of the algebraic equation. The next step would be to guess another value for x and then check this.

Here is an alternative to find the value(s) of x that satisfy the algebraic expression. This method is based on realizing that $x^2 - 2x + 1$ is a "perfect square:"

$x^2 - 2x + 1 = (x - 1)^2$.

So, the algebraic expression may be written as:

$(x - 1)^2 = 0$.

Taking the square root of each side of this:

$x - 1 = 0$,

so $x = 1$. (See Appendix F for more information on solving linear equations.)

The crucial step in the second approach of Example C.1 was recognizing that $x^2 - 2x + 1$ was the same as the product of factors: $(x - 1)(x - 1)$. The operation of converting the "expanded" expression, $x^2 - 2x + 1$, into a product of two factors is called "factoring" the algebraic expression.

Not every algebraic expression can be factored, and factoring is not always a straight-forward process. In the remainder of this appendix, we will outline four strategies that can help you to factor algebraic expressions.

Strategy 1: Common factors

Often, all terms in an expression will have a common factor. A useful simplification can be to extract this factor from each term in the expression. This operation can be thought of as the distributive law in reverse.

Example C.2

Factor each of the following expressions. If you are able to find a common factor, say what that factor is.

a) $x^{3/2} + x + \sqrt{x}$.

b) $e^{2t} + e^{t} + e^{\sin(t) + 1}$.

c) $y^3 + 2y^2 + y$.

Solution:

a) $x^{3/2} + x + \sqrt{x} = \sqrt{x} \cdot (x + \sqrt{x} + 1)$. The common factor is $\sqrt{x}$.

b) $e^{2t} + e^{t} + e^{\sin(t) + t} = e^{t} \cdot (e^{t} + 1 + e^{\sin(t)})$. The common factor is e^{t}.

c) $y^3 + 2y^2 + y = y \cdot (y^2 + 2y + 1)$. The common factor is y.

Strategy 2: Grouping like terms

Many algebraic expressions do not have a common factor that is shared by all terms. However, some of the terms may have a common factor. It can be useful to group these "like" terms and extract the common factor from them.

Example C.3

Factor each of the following expressions.

a) $e^{2x} + x^2 + x \cdot e^{x}$.

b) $e^{2x} + x^2 + 2 \cdot x \cdot e^{x}$.

c) $A(1 + e^{x}) + A(1 + e^{x}) \cdot e^{x}$.

Solution:

a) $e^{2x} + x^2 + x \cdot e^{x} = e^{x} \cdot (e^{x} + x) + x^2 = e^{2x} + x \cdot (x + e^{x})$.

b) $e^{2x} + x^2 + 2 \cdot x \cdot e^{x} = e^{2x} + x \cdot e^{x} + x^2 + x \cdot e^{x} = e^{x} \cdot (e^{x} + x) + x \cdot (x + e^{x}) = (e^{x} + x) \cdot (e^{x} + x)$.

c) $A(1 + e^{x}) + A(1 + e^{x}) \cdot e^{x} = A(1 + e^{x}) \cdot (1 + e^{x})$.

Strategy 3: Perfect squares

In Appendix B, several special cases of multiplying brackets were noted. Two of these are the "perfect squares:"

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$.

Example C.4

Factor each of the following expressions.

a) $r^2 - 4r + 4$.

b) $x + 2\sqrt{x} + 1$.

c) $2^{2x} + 2^{x+1} + 1$.

Solution:

a) $r^2 - 4r + 4 = (r + 2)^2$.

b) $x + 2\sqrt{x} + 1 = \left(\sqrt{x} + 1\right)^2$.

c) $2^{2x} + 2^{x+1} + 1 = (2^x)^2 + 2 \cdot 2^x + 1 = (2^x + 1)^2$.

Strategy 4: Differences of squares

The last special case of multiplying brackets from Appendix B was the "difference of two squares:"

$(a + b)(a - b) = a^2 - b^2$.

Example C.5

Factor each of the following expressions.

a) $t^2 - 81$.

b) $t^4 - 16$.

c) $\sin^4(x) - \cos^4(x)$.

Solution:

a) $t^2 - 81 = (t - 9)(t + 9)$

b) $t^4 - 16 = (t^2 - 4)(t^2 + 4) = (t - 2)(t + 2)(t^2 + 4)$.

c) $\sin^4(x) - \cos^4(x) = (\sin^2(x) - \cos^2(x))(\sin^2(x) + \cos^2(x)) = (\sin(x) - \cos(x))(\sin(x) + \cos(x))$.

Note that in Part (c), the trigonometric identity, $\cos^2(x) + \sin^2(x) = 1$ was used to simplify the expression.

Factoring Quadratic Expressions

As indicated by Example C.1, being able to factor quadratic expressions can be a very useful tool for solving equations. In Appendix H you will use inequalities to determine the sign of an algebraic expression. In such a situation, factoring the algebraic expression can also help with the mathematical analysis (see Example H.4).

Example C.6

Factor: $r^2 + 12r + 32$.

Solution:

We are trying to find two numbers, say a and b, so that:

$(r + a)(r + b)$

will multiply out to give: $r^2 + 12r + 32$. From Appendix B, $(r + a)(r + b)$ multiplies out to give:

$(r + a)(r + b) = r^2 + ar + br + ab = r^2 + (a + b)r = ab$.

Comparing this algebraic expression with $r^2 + 12r + 32$, you are looking for a and b so that:

$a + b = 12$, and,

$ab = 32$.

Two numbers that do this are: $a = 8$ and $b = 4$. Thus:

$r^2 + 12r + 32 = (r + 8)(r + 4)$.

Example C.7

Factor: $e^{2t} + 3e^t + 2$.

Solution:

The laws of exponents from Appendix A give that: $e^{2t} = (e^t)^2$. Using this, the algebraic expression that you have to factor begins to resemble a quadratic expression:

$e^{2t} + 3e^t + 2 = (e^t)^2 + 3e^t + 2$.

If you write r instead of e^t, this expression looks just like a quadratic expression, which can be factored just as in Example C.6:

$e^{2t} + 3e^t + 2 = (e^t)^2 + 3e^t + 2 = r^2 + 3r + 2 = (r + 1)(r + 2)$.

Converting back by substituting e^t for r gives:

$e^{2t} + 3e^t + 2 = (e^t + 1)(e^t + 2)$.

Exercises for Appendix C

For Problems 1-15, factor the quantity as much as possible.

1. $x - x^2 + 1$.

2. $x^2 - 3x + 2$.

3. $x \cdot \ln(x) - x$.

4. $x \cdot \sin(x) + x \cdot \cos(x)$.

5. $c^2 x + d^2 y$.

6. $2x + 4y^2$.

7. $a^2 - 4$.

8. $A \cdot e^{2t} + A \cdot t \cdot e^{2t} + A \cdot t^2 \cdot e^{2t}$.

9. $p(p + q) - q(p + q)$.

10. $(1 + 2t)^4 - w^2$.

11. $a^2 + 2ac + 2c^2$.

12. $4x + 16$.

13. $1 - \cos^2(u)$.

14. $Ar^2(1 + r) + 2A(1 + r)r + A(1 + r)$.

15. $x^2 + h^2 + 2hx - h^2$.

For Problems 16-20, factor the quadratic expression.

16. $x^2 + 6x + 8$.

17. $3x^2 + 9x + 6$.

18. $e^{4t} + 2e^{2t} + 1$.

19. $y^2 + 7y + 12$.

20. $t^2 + t + 2$.

Answers to Exercises for Appendix C

1. $x(1 - x) + 1$.

2. $(x - 1)(x - 2)$.

3. $x(\ln(x) - 1)$.

4. $x(\sin(x) + \cos(x))$.5. Without assuming anything about c and d, there is no way to factor this that provides obvious simplifications.

6. $2(x + 2y^2)$.

7. $(a + 2)(a - 2)$.

8. $A \cdot e^{2t}(1 + t + t^2)$.

9. $(p + q)(p - q)$.

10. $((1 + 2t)^2 + w)((1 + 2t)^2 - w)$.

11. $a(a + 2ac) + 2c^2$, or $a^2 + 2c(a + c)$, or $a(a + c) + c(a + 2c)$.

12. $4(x + 4)$.

13. $(1 + \cos(u))(1 - \cos(u))$.

14. $A(1 + r)^3$.

15. $x(x + 2h)$.

16. $(x + 2)(x + 4)$.

17. $3(x + 1)(x + 2)$.

18. $(e^{2t} + 1)^2$.

19. $(y + 3)(y + 4)$.

20. This does not factor. To see this, $b^2 - 4ac = 1 - 8 = -7$, which is negative.