Appendix D: Completing the Square and the Quadratic Formula

Factoring quadratic expressions such as:

$x^2 + 6x + 8$

was one of the topics introduced in Appendix C. Factoring quadratic expressions is a useful skill that can help you to find the solutions of equations. However, quadratics are not always easy to factor. Sometimes quadratics cannot be factored completely. Two procedures that can help you to factor a quadratic are "completing the square" and the quadratic formula.

Completing the square

In Appendix A, two special cases of expanding brackets were considered:

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$.

These were called "perfect squares." In Appendix C, two strategies suggested for factoring algebraic expressions were to look for perfect squares and to look for differences of squares.

Example D.1

Factor the quadratic expression:

$x^2 + 6x + 8$

by looking for perfect squares and differences of squares.

Solution:

The given algebraic expression is very close to the perfect square:

$(x + 3)^2 = x^2 + 6x + 9$.

So:

$x^2 + 6x + 8 = (x^2 + 6x + 9) - 1 = (x + 3)^2 - 1$.

Since $1 = 1^2$, the last expression is a difference of two squares, and so:

$x^2 + 6x + 8 = (x + 3)^2 - 1^2 = (x + 3 +1)(x + 3 - 1) = (x + 4)(x - 2)$.

Example D.1 is a very elaborate and counter-intuitive way to factor a straight-forward algebraic expression like $x^2 + 6x + 8$. The process outlined in Example D.1 has one great advantage over quicker, more intuitive methods: the method of Example D.1 will work when the numbers involved are more complicated and it is harder to factor the quadratic expression.

Example D.2

Factor the quadratic expression:

$x^2 + 66x + 8$

by looking for perfect squares and differences of squares.

Solution:

The perfect square that this quadratic expression is related to is:

$(x + 33) = x^2 + 66x + 1089$.

The quadratic expression that we have to factor is related to this perfect square:

$x^2 + 66x + 8 = (x^2 + 66x + 1089) - 1081 = (x + 33)^2 - 1081$.

This last algebraic expression is a difference of two squares, so:

$x^2 + 66x + 8 = (x + 33 + 32.879)(x + 33 - 32.879) = (x + 65.879)(x + 0.121)$.

The number 32.879 appears because the square root of 1081 is approximately 32.879.

The process of factoring quadratic equations illustrated in Examples D.1 and D.2 is called "completing the square." As indicated, not all quadratics can be completely factored. The process of completing the square can tell you when this is the case.

Example D.3

Try to factor the quadratic expression:

$x^2 + 6x + 10$

by looking for perfect squares and differences of squares.

Solution:

Following the pattern set out in Examples D.1 and D.2, you could start by finding a perfect square that is closely related to $x^2 + 6x + 10$. The perfect square:

$(x + 3)^2 = x^2 + 6x + 9$

is very closely related to $x^2 + 6x + 10$. In fact,

$x^2 + 6x + 10 = (x^2 + 6x + 9) + 1 = (x + 3)^2 + 1$.

This is where the pattern established in Examples D.1 and D.2 breaks down, because the expression that we have is not a difference of two squares. The expression:

$(x + 3)^2 + 1$

cannot be factored any further because it is the sum rather than the difference of two squares.

The Quadratic Formula

The process of factoring a quadratic expression by completing the square can be summarized as an algebraic formula. The formula is this:

If you are trying to factor the quadratic expression:

$ax^2 + bx + c$,

then the factors are:

$$a\left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right).$$

There is one important caveat when using this method to factor a polynomial: because you have to take a square root, the quantity:

$b^2 - 4ac$

must be greater than or equal to zero. If $b^2 - 4ac$ is negative, then it is not possible to factor the quadratic expression $ax^2 + bx + c$.

Example D.4

Use the quadratic formula to factor the following quadratic expressions. If it is not possible to factor any of the quadratic expressions, indicate why you think this to be the case.

a) $6x^2 + 9x + 2$.

b) $x^2 + x + 1$.

c) $x^2 + 2x + 1$.

Solution:

a) To factor $6x^2 + 9x + 2$, note the similarities between this expression and $ax^2 + bx + c$. The correspondence is that:

$a = 6$.
$b = 9$.

$c = 2$.

Before plugging into the formula, it is wise to make sure that the quantity $b^2 - 4ac$ is greater than or equal to zero. (If the quantity $b^2 - 4ac$ is negative, then the quadratic expression cannot be factored, so there will be no sense in trying.)

$b^2 - 4ac = 9^2 - 4{\cdot}6{\cdot}2 = 81 - 48 = 33$.

The number 33 is greater than or equal to zero, so the quadratic will factor. Plugging $a = 6$, $b = 9$ and $c = 2$ into the formula gives:

$6x^2 + 9x + 2 = 6(x + 1.2287)(x + 0.2713)$.

b) To factor $x^2 + x + 1$, again note the similarities between this expression and $ax^2 + bx + c$. In this case, the correspondence is that:

$a = 1$.
$b = 1$.
$c = 1$.

To check whether or not the expression $x^2 + x + 1$, can be factored, you can check the sign of $b^2 - 4ac$. In this case,

$b^2 - 4ac = 1 - 4{\cdot}1{\cdot}1 = 1 - 4 = -3$.

Since $b^2 - 4ac$ is negative, the quadratic expression $x^2 + x + 1$ cannot be factored.

c) Here the analysis is just like the previous two cases, except that:

$a = 1$.
$b = 2$.
$c = 1$.

Checking the quantity $b^2 - 4ac$ gives

$b^2 - 4ac = 4 - 4{\cdot}1{\cdot}1 = 0$.

As this is not negative, the quadratic expression $x^2 + 2x + 1$ will factor. Plugging $a = 1$, $b = 2$ and $c = 1$ into the formula gives:

$(x + 1)(x + 1) = (x + 1)^2$.

An alternative (and equally valid) way to factor this particular quadratic expression would have been to realize that $x^2 + 2x + 1$ was a perfect square, so that you can determine that $x^2 + 2x + 1 = (x + 1)^2$ without having to use the formula.

Using Completing the Square to Obtain the Quadratic Formula

You might wonder how the formula for the factors of the quadratic expression $ax^2 + bx + c$ was obtained. The working shown below indicates how the process of completing the square may be used to obtain the formula. The working presented below is quite formidable because it features a lot of symbols, rather than just concrete numbers. Use the explanatory notes for each step to follow what is going on.

$ax^2 + bx + c$

$= \; a\left(x^2 + \frac{b}{a}x + \frac{c}{a} \right)$ Factor out the a.

$= \; a\left(x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} \right)$ Make the expression as much like the perfect square: $\left[x + \frac{b}{2a} \right]^2$ as possible.

$= \; a\left(\left[x + \frac{b}{2a} \right]^2 - \left[\left(\frac{b}{2a}\right)^2 - \frac{c}{a} \right] \right)$ Factor the perfect square and group the left-over terms together.

$= \; a\left(\left[x + \frac{b}{2a} \right]^2 - \left[\frac{b^2}{4a^2} - \frac{c}{a} \right] \right)$ Simplify the term: $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$.

$= \; a\left(\left[x + \frac{b}{2a} \right]^2 - \frac{b^2 - 4ac}{4a^2} \right)$ Put the two terms over a common denominator. (See Appendix E.) Realize that what you have is a difference of two squares.

$= \; a\left(x + \frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{4a^2}} \right)\left(x + \frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{4a^2}} \right)$ Use the difference of squares to factor.

$= \; a\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \right)$ Take the square roots of the numerators and denominators.

$= \; a\left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a} \right)\left(x + \frac{b - \sqrt{b^2 - 4ac}}{2a} \right).$ Combine like terms.

Exercises for Appendix D

For Problems 1-10, complete the square for the given quadratic expressions.

1. $y^2 + 2y$.

2. $2a^2 + 8a + 5$.

3. $u^2 - 14u$.

4. $x^4 + 4x^2 + 1$.

5. $3r^2 + 12r + 16$.

6. $-x^2 + 10x + 1$.

7. $t^2 - 7t - 8$.

8. x^2.

9. $3r^2 - 24r + 14$.

10. $w^2 + 3w$.

For Problems 11-15, factor the quadratic expressions (if possible).

11. $r^2 + 7r + 12$.

12. $2y^2 + 5y + 3$.

13. $e^{2x+1} + 4e^{x+1} + 4e$.

14. $3t^2 + 13t + 9$.

15. $-u^2 + 4u - 5$.

Answers to Exercises for Appendix D

1. $(y + 1)^2 - 1$.
2. $2(a + 2)^2 - 3$.
3. $(u - 7)^2 - 49$.
4. $(x^2 + 2)^2 - 3$.
5. $3(r + 2)^2 + 4$.
6. $-(x - 5)^2 + 24$.
7. $(t - 3.5t) - 20.25$.
8. x^2.
9. $3(r - 4)^2 - 34$.

10. $(w + 3/2)^2 - 2.25$.

11. $(r + 3)(r + 4)$.

12. $2(y + 1)(y + 3/2)$.

13. $e(e^x + 2)^2$.

14. $3(t + 3.468)(t + 0.865)$.

15. This quadratic does not factor. To see this, $b^2 - 4ac = 16 - 4(-1)(-5) = -4$.