Appendix E: Manipulating Fractions

The rules for manipulating fractions that involve algebraic expressions are exactly the same as the rules for manipulating fractions that involve numbers.

The fundamental rules for combining and manipulating fractions are listed below. The uses of these rules are illustrated more completely later in this appendix.

a) Adding fractions

$$\frac{a}{c} + \frac{b}{d} = \frac{a \cdot d + b \cdot c}{c \cdot d}$$

b) Subtracting fractions

$$\frac{a}{c} - \frac{b}{d} = \frac{a \cdot d - b \cdot c}{c \cdot d}$$

c) Multiplying fractions

$$\frac{a}{c} \cdot \frac{b}{d} = \frac{a \cdot b}{c \cdot d}$$

d) Dividing fractions

$$\frac{a/c}{b/d} = \frac{a}{c} \cdot \frac{d}{b} = \frac{a \cdot d}{b \cdot c} \qquad \text{"Invert and multiply."}$$

Numerator and denominator

Fractions express a ratio of two quantities. For example, the fraction

$$\frac{a}{b}$$

expresses the ratio of quantity a to quantity b. The quantity that appears on the top of the fraction is called the numerator. In this case, the numerator is a. The quantity that appears on the bottom of the fraction is called the denominator. In this case, the denominator is b.

Finding a common denominator

Adding and subtracting fractions usually requires a common denominator, that is, all of the fractions involved have the same denominator.

Example E.1

Find common denominators for the following collections of fractions. Express the fractions using this common denominator.

a) $\dfrac{1}{2}, \dfrac{1}{x}$.

b) $\dfrac{1}{x^2 - x}, \dfrac{1}{x}$.

c) $\dfrac{1}{x^2 + 2x + 1}, \dfrac{1}{x + 1}$.

Solution:

a) The common denominator is $2x$. The two fractions can be expressed by multiplying both numerator and denominator (top and bottom) by whatever factor is needed to convert the denominator to the common denominator.

$$\frac{1}{2} = \frac{1}{2} \cdot \frac{x}{x} = \frac{x}{2x}$$

$$\frac{1}{x} = \frac{1}{x} \cdot \frac{2}{2} = \frac{2}{2x}.$$

b) A common denominator can always be obtained by just multiplying each of the denominators together. A possible common denominator is

$$x \cdot (x^2 - x) = x^3 - x^2.$$

Expressing the two fractions using this common denominator:

$$\frac{1}{x^2 - x} = \frac{1}{x^2 - x} \cdot \frac{x}{x} = \frac{x}{x^3 - x^2}$$

$$\frac{1}{x} = \frac{1}{x} \cdot \frac{x^2 - x}{x^2 - x} = \frac{x^2 - x}{x^3 - x^2}.$$

Note, however, that $x^2 - x = x \cdot (x - 1)$. The significance of this observation is that since $x^2 - x$ already has the denominators of both $\frac{1}{x^2 - x}$ and $\frac{1}{x}$ as factors. So, $x^2 - x$ is already a common denominator for both of these fractions and could be used instead of $x^3 - x^2$.

c) One way to obtain a common denominator is to just multiply the denominators of the two fractions

$$\frac{1}{x^2 + 2x + 1}, \frac{1}{x + 1}$$

together. However, $x^2 + 2x + 1 = (x + 1)^2$. Since $x + 1$ is a factor of $x^2 + 2x + 1$, $x^2 + 2x + 1$ can be used as a common denominator. Expressing the two fractions using the common denominator $x^2 + 2x + 1$:

$$\frac{1}{x^2 + 2x + 1}$$

$$\frac{1}{x+1} = \frac{1}{x+1} \cdot \frac{x+1}{x+1} = \frac{x+1}{x^2+2x+1}.$$

The advantage of using $x^2 + 2x + 1$ as a common denominator (rather than, say, the more obvious $(x+1)\cdot(x^2+2x+1) = x^3 + 3x^2 + 3x + 1$) is that the fractions that you obtain using the common denominator are usually the simplest possible.

Adding and subtracting fractions

Fractions can only be added and subtracted when the two fractions have the same denominated. If the two fractions do not have the same denominator, then a common denominator must be found before the fractions can be added or subtracted. The necessity of finding a common denominator is why the product of the two denominators (i.e. $b\cdot d$) appears in the rules expressed below. Expressed using algebraic symbols, the rules for adding and subtracting functions are:

$$\frac{a}{c} + \frac{b}{d} = \frac{a}{c} \cdot \frac{d}{d} + \frac{b}{d} \cdot \frac{c}{c} = \frac{a\cdot d + b\cdot c}{c\cdot d}$$

$$\frac{a}{c} - \frac{b}{d} = \frac{a}{c} \cdot \frac{d}{d} + \frac{b}{d} \cdot \frac{c}{c} = \frac{a\cdot d - b\cdot c}{c\cdot d}.$$

Example E.2

Evaluate and simplify each of the algebraic expressions.

a) $\dfrac{1}{x+h} - \dfrac{1}{x}.$

b) $\dfrac{(x+h)^2}{h} - \dfrac{x}{h}.$

c) $\dfrac{\sqrt{x}}{x+1} + \dfrac{x+1}{\sqrt{x}}.$

d) $\dfrac{r+1}{r^2+5r+6} + \dfrac{r+2}{r^2+4r+3}.$

Solution:

a) $\dfrac{1}{x+h} - \dfrac{1}{x} = \dfrac{x-(x+h)}{x(x+h)} = \dfrac{-h}{x^2+xh}.$

b) $\dfrac{(x+h)^2}{h} - \dfrac{x}{h} = \dfrac{x^2+2hx+h^2-x}{h} = \dfrac{x^2+(2h-1)x+h^2}{h}.$ (The two fractions already have the same denominator.)

c) $\dfrac{\sqrt{x}}{x+1}+\dfrac{x+1}{\sqrt{x}}=\dfrac{\left(\sqrt{x}\right)\cdot\left(\sqrt{x}\right)+(x+1)\cdot(x+1)}{(x+1)\cdot\sqrt{x}}=\dfrac{x+x^2+2x+1}{(x+1)\cdot\sqrt{x}}=\dfrac{x^2+3x+1}{(x+1)\cdot\sqrt{x}}.$

d) $\dfrac{r+1}{r^2+5r+6}+\dfrac{r+2}{r^2+4r+3}=\dfrac{(r+1)}{(r+2)(r+3)}+\dfrac{(r+2)}{(r+1)(r+3)}=\dfrac{(r+1)^2+(r+2)^2}{(r+1)(r+2)(r+3)}.$

Some further simplifications are possible in Part (d). For example, you could multiply out all of the brackets. The important point in Part (d), however, is that by using the simplest (or "least") common denominator, the algebraic form of the result will be as simple as possible. In the case of Part (d), the simplest common denominator is $(r + 1)(r + 2)(r + 3)$, rather than the more obvious common denominator $(r + 1)(r + 2)(r + 2)(r + 3)$.

Multiplying fractions

Multiplying two fractions is perhaps the most straight-forward of all operations. You simply multiply the numerators and multiply the denominators. Expressed using algebraic symbols, this rule is:

$$\frac{a}{c}\cdot\frac{b}{d}=\frac{a\cdot b}{c\cdot d}.$$

Example E.3

Evaluate and simplify the following fractions as much as possible.

a) $\dfrac{2y}{y+1}\cdot\dfrac{\sqrt{y}}{y+2}.$

b) $\dfrac{\sin(\theta)+\cos(\theta)}{1+\theta}\cdot\dfrac{\sin(\theta)+\cos(\theta)}{1-\theta}.$

Solution:

a) $\dfrac{2y}{y+1}\cdot\dfrac{\sqrt{y}}{y+2}=\dfrac{2y\cdot\sqrt{y}}{(y+1)\cdot(y+2)}=\dfrac{2y^{3/2}}{y^2+3y+2}.$

b) $\dfrac{\sin(\theta)+\cos(\theta)}{1+\theta}\cdot\dfrac{\sin(\theta)+\cos(\theta)}{1-\theta}=\dfrac{\sin^2(\theta)+2\sin(\theta)\cos(\theta)+\cos^2(\theta)}{1-\theta^2}.$

Dividing fractions

Many people remember the rule for dividing one fraction by another by remembering that you must "invert the denominator and multiply." Expressed as algebraic symbols, the rule is:

$$\frac{a/_c}{b/_d} = \frac{a}{c} \cdot \frac{d}{b} = \frac{a \cdot d}{b \cdot c}.$$

Example E.4

Evaluate or simplify each of the fractions given below.

a) $\dfrac{x/_2}{(x+2)/_4}$.

b) $\dfrac{t}{t^{-1/2}}$.

c) $\dfrac{\dfrac{2r}{\sqrt{1+r^2}}}{\dfrac{1+r}{1-r^2}}$.

Solution:

a) $\dfrac{x/_2}{(x+2)/_4} = \dfrac{x}{2} \cdot \dfrac{4}{x+2} = \dfrac{2x}{x+2}.$

b) $\dfrac{t}{t^{-1/2}} = \dfrac{t}{\dfrac{1}{\sqrt{t}}} = t \cdot \dfrac{\sqrt{t}}{1} = t^{3/2}.$

c) $\dfrac{\dfrac{2r}{\sqrt{1+r^2}}}{\dfrac{1+r}{1-r^2}} = \dfrac{2r}{\sqrt{1+r^2}} \cdot \dfrac{1-r^2}{1+r} = \dfrac{2r}{\sqrt{1+r^2}} \cdot \dfrac{1-r}{1} = \dfrac{2r-r^2}{\sqrt{1+r^2}}.$

Simplifying complicated fractions

Sometimes, fractions will have other fractions in their numerator or denominator (or both). To simplify and evaluate complicated fractions, evaluate the numerator and denominator separately , and then divide the two.

Example E.5

Simplify each of the complicated fractions as much as possible.

a) $\dfrac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h}$.

b) $\dfrac{(x+y)^{-1/2} + \sqrt{x+y}}{x+y}$.

c) $\dfrac{\dfrac{r}{r+1} + \dfrac{1}{r}}{\dfrac{r^2}{9-r} + \dfrac{r}{9-r}}$.

Solution:

a) $\dfrac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \dfrac{\dfrac{x-(x+h)}{x\cdot(x+h)}}{h} = \dfrac{\dfrac{-h}{x\cdot(x+h)}}{h} = \dfrac{-h}{h\cdot x\cdot(x+h)} = \dfrac{-1}{x\cdot(x+h)}$.

b)

$\dfrac{(x+y)^{-1/2} + \sqrt{x+y}}{x+y} = \dfrac{\dfrac{1}{\sqrt{x+y}} + \sqrt{x+y}}{x+y} = \dfrac{\dfrac{1+\left(\sqrt{x+y}\right)\left(\sqrt{x+y}\right)}{\sqrt{x+y}}}{x+y} = \dfrac{1+x+y}{(x+y)\cdot\sqrt{x+y}} = \dfrac{1+x+y}{(x+y)^{3/2}}$

.

c) $\dfrac{\dfrac{r}{r+1} + \dfrac{1}{r}}{\dfrac{r^2}{9-r} + \dfrac{r}{9-r}} = \dfrac{\dfrac{r^2+1}{r\cdot(r+1)}}{\dfrac{r^2+r}{9-r}} = \dfrac{r^2+1}{r\cdot(r+1)} \cdot \dfrac{9-r}{r^2+r} = \dfrac{\left(r^2+1\right)\cdot(9-r)}{r\cdot(r+1)\cdot\left(r^2+r\right)} = \dfrac{-r^3+9r^2-r+9}{r^4+2r^3+r^2}$.

Exercises for Appendix E

For Problems 1-10, evaluate the quantity without using a calculator.

1. $\dfrac{2}{7} + \dfrac{3}{9}$.

2. $\dfrac{1}{2} - \dfrac{1}{3}$.

3. $\dfrac{-1}{4} + \dfrac{1}{2}$.

4. $\dfrac{2}{3y} + \dfrac{y}{8}$.

5. $\dfrac{2a}{x} - \dfrac{3}{7x}$.

6. $\dfrac{2}{7} \cdot \dfrac{a}{9b}$.

7. $\dfrac{\frac{7}{x}}{x^2}$.

8. $\dfrac{\frac{9}{b^2}}{\frac{27}{b^3}}$.

9. $\dfrac{3z^2}{\frac{2z}{11}}$.

10. $\dfrac{3+y}{3-y} + \dfrac{3}{y-3}$.

For Problems 11-20, perform the operation indicated. Simplify your answers as much as possible.

11. $L + w + \dfrac{L}{L+w}$.

12. $a - \dfrac{b}{a - \dfrac{b}{a}}$.

13. $\dfrac{(x+h)^2 - (x-h)^2}{2h}$.

14. $\dfrac{2+b^{-1}}{7+b^{-2}}$.

15. $\dfrac{q^2 - 1}{q^2 + 2q + 1}$.

16. $\dfrac{2}{7} + \dfrac{x}{x-4}$.

17. $\dfrac{1}{\sqrt{t}} + \dfrac{t^{3/2}}{\sqrt{t} - \frac{1}{\sqrt{t}}}$.

18. $\dfrac{\frac{x^2+2xy+y^2}{x}}{x^2+2xy+y^2}$.

19. $\dfrac{a^{-2}+a^{-3}}{1+a^{-1}}$.

20. $\dfrac{1+2\cdot\sin(\theta)\cdot\cos(\theta)}{\sin(\theta)+\cos(\theta)}$.

For Problems 21-25, decide whether each of the statements are true or false.

21. $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{2}{x+y}$.

22. $\dfrac{x+y}{y}=1+\dfrac{x}{y}$.

23. $\dfrac{y}{x+y}=1+\dfrac{y}{x}$.

24. $\dfrac{\frac{2}{x}}{x^2}=2x$.

25. $\dfrac{2a^2+2ab+b^2}{a^2+2ab+b^2}=1+a^2$.

Answers to Exercises for Appendix E

1. 13/21.
2. 1/6.
3. 1/4.
4. (16 + 3*y*²)/(24*y*).
5. (14*a* - 3)/(7*x*).
6. (2*a*)/(63*b*).
7. 7/(*x*³).
8. *b*/3.
9. (33*z*)/2.
10. *y*/(3 - *y*).
11. (*L*² + 2*wL* + *w*² + *L*)/(*L* + *w*).

12. $a(a^2 - 2b)/(a^2 - b)$.

13. $2x$.

14. $b(2b + 1)/(7b^2 + 1)$.

15. $(q - 1)/(q + 1)$.

16. $(9x - 8)/(7x - 28)$.

17. $(t^2 + t^{1/2} + t^{-1/2})/(t - 1)$.

18. $1/x$.

19. $1/a^2$.

20. $\sin(\theta) + \cos(\theta)$.

21. False.

22. True.

23. False.

24. False.

25. False.