Appendix G: Solving Systems of Equations

Often, you will be able to solve a problem that you are interested in by finding a solution for a single equation. More complicated problems may require you to solve more than one equation.

Example G.1

A shoe company manufactures running shoes and walking shoes. Each day, the company orders in enough materials to make 80 pairs of shoes, and has 100 person-hours of labor available. Running shoes and walking shoes take the same amount of materials each. Each pair of walking shoes takes 1 hour assemble, and each pair of running shoes takes 2 hours to assemble.

a) Represent this information using one or more equations.

b) If the company wants to use all of the materials and all of the labor each day, how many pairs of running and walking shoes should they manufacture?

Solution:

a) Let R represent the number of pairs of running shoes that the company makes each day and W represent the number of pairs of walking shoes that the company makes each day.

The company has enough materials to make 80 pairs of shoes, or in other words the number of pairs of running shoes plus the number of pairs of walking shoes should add up to ninety. Expressing this as an equation:

$R + W = 80.$

Each pair of running shoes takes two hours to assemble, and each pair of walking shoes takes one hour to assemble. Each day, there are 100 hours of labor available, so:

$2R + W = 100.$

b) In order to find the number of pairs of each type of shoe that the company should manufacture, we have to find a numerical value of R and a numerical value of W. We want to find numerical values that satisfy both of the equations:

$R + W = 80$ ... (1)

and,

$2R + W = 100.$ ... (2)

One way to proceed is to make W the subject of Equation (1),

$W = 80 - R$

and to substitute this for W in Equation (2):

$2R + (80 - R) = 100.$

This gives an equation that only involves R, and allows the numerical value for R to be determined:

$2R + (80 - R) = 100.$

$R = 100 - 80.$

$R = 20.$

Knowing the numerical value for R, you can substitute this back into either Equation (1) or Equation (2) to find the numerical value of W:

$20 + W = 80.$

$W = 80 - 20.$

$W = 60.$

The final conclusion to this problem is that the company should manufacture 20 pairs of running shoes and 60 pairs of walking shoes.

The collection of equations that you wish to solve is called a system of equations. In order to be certain that you will obtain a solution, you need to have the same number of equations as variables. In Example G.1, there were two variables (R and W) and two equations (one for materials and one for labor). In Example G.1, all of the equations were linear equations. The equations in a system of equations will often be linear equations, but they do not have to be.

Example G.2

Strontium-90 is a radioactive substance. Strontium-90 is sometimes released during nuclear accidents and is regarded as a very dangerous substance because it becomes incorporated into bone tissue.

The mass, M, of strontium-90 remaining in a person's bones after T years is described by an exponential decay formula:

$M = M_0 \cdot B^T$,

where M_0 is the mass of strontium-90 that was absorbed into the person's bones when they were exposed to radioactivity, and B is a number.

Bone samples from the person indicated that 12 years after exposure, the person had 0.75 grams of strontium-90 in their body, and 30 years after exposure, the person had 0.488 grams of strontium-90 in their body.

a) What is the value of the constant B ?

b) How much strontium-90 did the person have in their body just after being exposed to the radiation?

Solution:

Before attempting to answer questions (a) and (b), it can be useful to identify the important information in the problem, and put it all together.

We are told that when $T = 12$, $M = 0.75$. Putting this together with the equation gives:

$0.75 = M_0 \cdot B^{12}$ ... (1)

Likewise, we are told that when $T = 30$, $M = 0.488$, so that:

$0.488 = M_0 \cdot B^{30}$ ... (2)

There are two equations (Equations (1) and (2)), and two unknowns, M_0 and B.

a) The strategy in Example G.1 was to try to combine the two equations in order to eliminate one of the variables. This can be accomplished here by division to cancel out the M_0's:

$$\frac{0.488}{0.75} = \frac{M_0 \cdot B^{30}}{M_0 \cdot B^{12}} = \frac{B^{30}}{B^{12}} = B^{30-12} = B^{18}.$$

(The Laws for Exponents were used to combine the powers of B - see Appendix A.) We are left with an equation that includes only one of the unknown quantities, B:

$B^{18} = 0.65$

$B = (0.65)^{1/18} = 0.9764.$

b) In Example G.1, once the numerical value of one of the quantities was known, the numerical value of the other quantity could be found by plugging back into one of the original equations. To find the numerical value of M_0, we can plug $B = 0.9764$ into Equation (1) and solve for M_0:

$0.75 = M_0 \cdot (0.9764)^{12}$

$$\frac{0.75}{(0.9764)^{12}} = M_0$$

$M_0 = 0.9989.$

So, just after exposure to the radiation, the person had 0.9989 grams of strontium-90 in their bones.

Exercises for Appendix G

For Problems 1-5, solve the system of linear equations.

1. $x + y = 2.$
   $x - y = -2.$

2. $3p + 2q = 10.$
$p - q = 0.$

3. $r - 2s = 1.$
$s + r = 4.$

4. $x = 8y + 9.$
$x + 2y = 29.$

5. $t = 4w + 9.$
$t = 37.$

For Problems 6-10, solve the system of equations.

6. $x^2 + y^2 = 1.$
$x - y = 0.$

7. $A \cdot e^{2k} = 10.$
$A \cdot e^{4k} = 20.$

8. $r = 4 - s^2.$
$s + r = 1.$

9. $y = 2^x.$
$y = 2x.$

10. $p = \dfrac{1}{q+1}.$
$p - q = 1.$

Answers to Exercises for Appendix G

1. $x = 0$ and $y = 2$.

2. $p = 2$ and $q = 2$.

3. $s = 1$ and $r = 3$.

4. $x = 25$ and $y = 2$.

5. $t = 37$ and $w = 7$.

6. There are two solutions: $(x, y) = (1/\sqrt{2}, 1/\sqrt{2})$ and $(x, y) = (-1/\sqrt{2}, -1/\sqrt{2})$.

7. $A = 5$ and $k = \ln(2)/2 \approx 0.34657359$.

8. This system of equations has no solutions.

9. There are two solutions: $(x, y) = (1, 2)$ and $(x, y) = (2, 4)$.

10. There are two solutions to this equation: $(p, q) = (1, 0)$ and $(p, q) = (-1, -2)$.